SMOOTH APPROXIMATIONS TO POLYHEDRA

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1. INTRODUCTION

This paper is concerned with relations between combinatorial and differential topology. In studying such relations, we work with piecewise smooth homeomorphisms restricted to subcomplexes of differentiable triangulations of euclidean n-space. This class of maps and spaces, although it is not a category, is appropriate because it contains the relevant piecewise linear and smooth categories, together with other spaces and maps useful in passing from one of these categories to the other.

1.1. *Vertical bars and carriers.* If V is a set of point sets, $|V|$ will denote the union of the elements of V. If these elements are disjoint, the *carrier* (*in* V) of a point $x \in |V|$ will mean the element of V containing x.

1.2. *Omitted modifiers.* In the foregoing and subsequent definitions, parentheses around a modifier indicate that it will sometimes be omitted for brevity.

We use K with or without indices to denote finite linear simplicial complexes in euclidean n-space E^n. Thus $|K|$ denotes a polyhedron. Since we use open simplexes, each point of $|K|$ has a carrier in K.

Dimensions are indicated, where relevant, by superscripts. The empty set $\emptyset$ has dimension -1. Except when we write K^{-1}, for the trivial complex $\{\emptyset\}$, K denotes an m-complex ($m \geq 0$).

A topological manifold is said to be *closed* if it is compact and has no boundary. A closed (n - 1)-manifold in E^n *surrounds* each subset of its interior.

Closed, half-open, and open directed line segments are denoted by [qp], [qp), (qp], and (qp); [qp] is a *vector*.

The terms *smooth, differentiable,* and *of class* C^∞ are used synonymously.

We depart from the piecewise smooth class for the sake of a result involving an (n - 1)-manifold of differentiability class C^1.

THEOREM I. *For each polyhedron* $|K| \subset E^n$ $(n > 1)$, *there exist a closed* C^1-(n - 1)-*manifold* M^{n-1} *and a set* J *of vectors such that*

(a) M^{n-1} *surrounds* $|K|$,

(b) *if* $[qp] \in J$, *then* $p \in M^{n-1}$, $q \in |K|$, *and* [pq] *is normal to* M^{n-1} *at* p,

(c) *the interior* N *of* M^{n-1} *satisfies the conditions*

$$\overline{N} = |J| \quad \textit{and} \quad \overline{N} - |K| = |\{[pq): [pq] \in J\}|$$

(the horizontal bar denotes closure),

Received August 20, 1966.

This work was supported by the National Science Foundation (Grants GP 4092 and GP 5610) and the Science Research Council of Great Britain. It is a revision of a paper written for the Topology Symposium of the University of Warwick, England, in 1966. Miss Keiko Kudo contributed to the revision.