ABELIAN ACTIONS ON 2-MANIFOLDS

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Dedicated to R. L. Wilder on his seventieth birthday.

Let G be a finite group. We shall consider the problem of classifying the effective actions of G on closed oriented surfaces $\mathfrak{M}$. If two such actions are equivalent, their orbit spaces are homeomorphic, and so it is natural to study the totality A of actions having a fixed orbit space M. The problem is (1) to determine the equivalence classes of A in the sense of putting them into one-to-one correspondence with the equivalence classes of some algebraic system and (2) to compute, using the algebraic scheme, the number of equivalence classes in A for different M's and G's. J. Nielsen [4] gave a solution of (1) for the case where G is cyclic, but he did not consider (2) explicitly. We give here a solution of (1) for $G = Z_p \times \cdots \times Z_p$ (p a prime) by showing that in this case the equivalence classes are in one-to-one correspondence with the equivalence classes of certain matrices over Z_p under certain operations on the columns. Actually, a solution for (1) can be given for arbitrary finite abelian groups. But in this more general case we have little information relative to (2), whereas in the case considered we do solve (2) in some simple instances.

A CLASSIFICATION THEOREM FOR FREE ACTIONS

A homeomorphism $X \to Y$ or $(X, x) \to (Y, y)$ is always "onto." A homeomorphism between oriented manifolds always preserves orientation.

1. We consider actions $a = (G, \mathfrak{X})$, where G is a fixed discrete group and $\mathfrak{X}$ a topological space. We denote by $\mu(a)$ the space in which the action a takes place: if $a = (G, \mathfrak{X})$, then $\mu(a) = \mathfrak{X}$. An action $(G, \mathfrak{X})$ is *effective* if the identity is the only element of G that leaves all points of $\mathfrak{X}$ fixed; it is *free* if no point of $\mathfrak{X}$ is left fixed by any element of $G - \{1\}$. Two actions $a = (G, \mathfrak{X})$ and $a' = (G, \mathfrak{X}')$ are *equivalent* (notation: $a \sim a'$) if there exists a homeomorphism $t\colon \mathfrak{X} \to \mathfrak{X}'$ such that $t(g\mathfrak{x}) = g(t\mathfrak{x})$ for $g \in G$, $\mathfrak{x} \in \mathfrak{X}$. To indicate that t defines an equivalence, we refer to it as an *equivalence map.*

A space X will be called *allowable* provided it is arcwise connected and semi-locally arcwise connected (so that the theory of coverings as described by paths is valid; see [2, pp. 89-97]), and provided further that it has the following "isomorphism replacement" property: if x, x' are distinct points of X, and u is a path in X from x to x' and u_π the isomorphism $\pi_1(X, x') \to \pi_1(X, x)$ induced by u, then there exists a homeomorphism $t\colon (X, x') \to (X, x)$ such that $t_\pi = u_\pi$, where t_π is the isomorphism of fundamental groups induced by t. A connected manifold admitting a differentiable structure is allowable. In particular, compact 2-manifolds are allowable.

Call an action $(G, \mathfrak{X})$ *allowable* if its orbit space X is allowable and if the pair $(\mathfrak{X}, \psi)$, where ψ is the natural map of $\mathfrak{X}$ onto X, is a covering of X.

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