WILD CELLS AND SPHERES IN HIGHER DIMENSIONS

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Dedicated to R. L. Wilder on his seventieth birthday.

1. INTRODUCTION

The purpose of this paper is to apply a theorem of Andrews and Curtis [1] to get a rapid formula for constructing wild k-cells and k-spheres in S^n. In Section 4 we construct an arc in S^n $(n > 3)$ that pierces no locally flat $(n - 1)$-sphere. (The somewhat lengthy interval between discovery and publication has led to the prior appearance of applications of and reference to this technique in the literature [9], [11].) Our starting point is the following obvious modification of the results of [1]:

THEOREM (Andrews and Curtis). *Let* α *be an arc in* S^n. *Then the suspension* $\sigma(S^n/\alpha)$ *of the quotient space* S^n/α *is homeomorphic to* S^{n+1}. (If X is compact, we use $\sigma(X)$ to denote the quotient space of $X \times [0, 1]$ obtained by pinching $X \times 0$ and $X \times 1$ to points.)

2. THE CONSTRUCTION α^*

Let α be an arc in S^n, and π the projection map $\pi\colon S^n \to S^n/\alpha$. This induces the natural suspensions $\sigma(\pi)\colon \sigma(S^n) \to \sigma(S^n/\alpha)$, where the image and domain spaces are both S^{n+1}. Let $\alpha^* = \sigma(\pi(\alpha)) \subset \sigma(S^n/\alpha)$ be the suspension of the point $\langle \alpha \rangle$ of S^n/α. Then α^* is an arc and $\sigma(\pi) \mid \sigma(S^n) - \sigma(\alpha)$ is a homeomorphism onto $\sigma(S^n/\alpha) - \alpha^*$. On the other hand, $\sigma(S^n) - \sigma(\alpha)$ is homeomorphic to $(S^n - \alpha) \times R'$, since $\sigma(\alpha)$ contains the suspension points. Hence

(2.1) $\sigma(S^n/\alpha) - \alpha^*$ *is homeomorphic to* $(S^n - \alpha) \times R'$,

(2.2) *for every arc* $\alpha \subset S^n$ *there is an arc* $\alpha^* \subset S^{n+1}$ *such that* $S^n - \alpha$ *and* $S^{n+1} - \alpha^*$ *have the same homotopy type,*

(2.3) *for each* $n \geq 3$ *there exists an arc in* S^n *whose complement is not simply connected.*

We get (2.3) by repeated applications of (2.2) to the arc (1.1) of [8].

(2.4) *For each pair* (n, k) *with* $n \geq 3$ *and* $1 \leq k \leq n$, *there exists a* k-*cell in* S^n *whose complement is not simply connected.*

Proof. Let P(n, k) denote the statement of (2.4) for a fixed admissible pair (n, k), and P(n, *) the statement for n fixed and all admissible k. P(3, *) is proved in [8]. Inductively, suppose P(n, *) is true. From (2.3) we have (n + 1, 1). But if $k > 1$, then P(n + 1, k) follows from P(n, k - 1). For if α^{k-1} is a (k - 1)-cell in S^n and $\pi_1(S^n - \alpha^{k-1})$ is nontrivial, then $\alpha^k = \sigma(\alpha^{k-1})$ is a k-cell in $S^{n+1} = \sigma(S^n)$. Since $\sigma(\alpha^{k-1})$ contains the suspension points, $S^{n+1} - \alpha^k$ is homeomorphic to $(S^n - \alpha^{k-1}) \times R'$.

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