EXTENSIONS OF ALGEBRA HOMOMORPHISMS

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The Hahn-Banach Theorem [1, pp. 28-29] has a variant, called the Monotone Extension Theorem [2, p. 20], [3, Corollary 2.3], in which the subadditive, nonnegatively homogeneous function of the former theorem is replaced by a partial-ordering condition. The Hahn-Banach Theorem has been generalized by Vincent-Smith [5] to the extent that the scalar ring of real numbers may be replaced by any ring of real-valued continuous functions on an extremely disconnected compact Hausdorff space. In a similar fashion the Monotone Extension Theorem can be so generalized. We cite the Hahn-Banach and Monotone Extension Theorems in this greater generality below. In this paper, we shall prove analogues of these two theorems for algebras and rings instead of modules. Our Theorem 1 is the analogue of the Monotone Extension Theorem for commutative algebras. We sharpen this result in Theorem 2 by replacing the requirement of commutativity by a weaker condition, and then we give a comparable result for rings instead of algebras (Theorem 3). Then, in Theorem 4, we convert the partial-ordering condition of Theorem 2 into a subadditive, nonnegatively homogeneous function, which yields a curious analogue of the Hahn-Banach Theorem for algebras instead of modules. Finally we give several applications of Theorems 2, 3, and 4.

Throughout the paper, R denotes the field of real numbers, X the partially ordered ring of real-valued continuous functions on some extremely disconnected compact Hausdorff space, and Y a subring of X. We consider X as an algebra over Y. We use the customary definitions of partially ordered modules, rings, and algebras, except that we do *not* require antisymmetry. An element a of a partially ordered module, ring, or algebra A is *positive* if $0 \leq a$, and the set of all positive elements of A (sometimes called the *positive wedge* of A) is denoted by A^+. A subset B of a partially ordered set A is *cofinal in* A if, for every $a \in A$, there exists some $b \in B$ such that $a \leq b$. If $\leq$ and $\leq'$ denote two partial orderings of a module, ring, or algebra A, then $\leq$ is called *finer than* $\leq'$ if $0 \leq' a$ implies $0 \leq a$, for all $a \in A$.

HAHN-BANACH THEOREM. *Let* B *be a submodule of a module* (A, +, .) *over* X, *and let* $P: A \to X$ *satisfy the conditions*

$$P(\alpha a) = \alpha P(a) \quad \textit{and} \quad P(a + b) \leq P(a) + P(b)$$

for all $\alpha \in X^+$ *and* $a, b \in A$. *If* $T: (B, +, .) \to (X, +, .)$ *is a homomorphism such that* $T \leq P$, *then there exists a homomorphism* $T^*: (A, +, .) \to (X, +, .)$ *that extends* T *and satisfies the inequality* $T^* \leq P$.

MONOTONE EXTENSION THEOREM. *Let* B *be a submodule of a partially ordered module* $(A, +, ., \leq)$ *over* Y *such that* B^+ *is cofinal in* A (*or if* $X = R$, B^+ *is cofinal in* A^+). *If* $T: (B, +, ., \leq) \to (X, +, ., \leq)$ *is a homomorphism, then there exists a homomorphism* $T^*: (A, +, ., \leq) \to (X, +, ., \leq)$ *that extends* T.

Since the proofs of these theorems do not require significantly more technique than is involved in the proof of the classical Hahn-Banach Theorem, we leave the proofs for the reader.

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