ON THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF $x'' + a(t)x = 0$

A. Meir, D. Willett, and J. S. W. Wong

The differential equation

$$x'' + a(t)x = 0 \qquad (t \geq 0) \tag{1}$$

has been widely investigated; see, for example, Cesari [2, pp. 80-90] and Bellman [1, Part 3]. In the special case when $a(t) > 0$ and

$$a(t) \uparrow \infty \quad \text{as } t \uparrow \infty, \tag{2}$$

Milloux [6], Hartman [4], and Prodi [7] have shown that (1) must have at least one nontrivial solution $x(t)$ such that

$$x(t) \to 0 \quad \text{as } t \to \infty. \tag{3}$$

Milloux [6] and Galbraith, McShane, and Parrish [3] have shown that in order that (3) holds for all solutions of (1), some additional condition on $a(t)$ besides (2) is needed. Furthermore, such a condition cannot be a simple restriction on the order of $a(t)$ or $a'(t)$, for Willett [9] has recently shown that for each $b(t) \geq 0$, there exists a function $a(t)$ such that $a'(t) \geq b(t)$ and (1) has a solution $x(t)$ with $\limsup_{t\to\infty} |x(t)| > 0$.

Sufficient conditions assuring that (3) holds for all solutions of (1) have been obtained (see, for example, [1, p. 88], [2, pp. 85-86], and [5]). One of the best of these conditions from a theoretical viewpoint is due to Sansone [8]: if $a(t)$ belongs to the class $C^1[0, \infty]$, then for every sequence $\{t_n\}$ subject to the conditions

$$t_n \to \infty, \qquad t_{n+1} - t_n \leq t_n - t_{n-1}, \qquad \limsup_{n\to\infty} \frac{t_{n+1} - t_n}{t_n - t_{n-1}} = 1,$$

it is true that

$$\sum_{n=1}^{\infty} (t_{n+1} - t_n) \min_{t_n \leq t \leq t_{n+1}} \frac{a'(t)}{a(t)} = \infty. \tag{4}$$

One of the reasons for the present paper is that for a given $a(t)$, conditions such as (4) are usually difficult to verify. It is our aim to present conditions on $a(t)$ that are usually easy to verify in practice, that imply that all solutions of (1) satisfy (3), and that in our opinion are quite general—for example, as general as Sansone's condition mentioned above. In its most general form, our condition requires the choice of a second function $q(t)$ that is in some sense a "smooth approximation" of $a(t)$. In many cases, the choice of $q(t)$ is obvious, and thus the condition is easily verified.

Let $\max(w(t), 0)$ be denoted by $w_+(t)$. Throughout the paper, we assume that $a(t)$ is of class $C^1[0, \infty)$ and that $a > 0$, $a' \geq 0$, and $a(t) \to \infty$ as $t \to \infty$.

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