PRIMITIVE INVARIANTS AND CONJUGATE CLASSES OF FUNDAMENTAL REPRESENTATIONS OF A COMPACT SIMPLY CONNECTED LIE GROUP

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Let G be a compact, connected, and simply connected Lie group. The rational cohomology algebra of G can be expressed as an exterior algebra

$$H^*(G; Q) = \Lambda_Q(x_1, \cdots, x_\ell),$$

where $\deg x_i = 2m_i - 1$ $(1 \leq i \leq \ell)$ and $\ell = \operatorname{rank} G$. The integers m_i are called the *primitive invariants* of G.

Among the complex representations of G there are ℓ fundamental ones, $\rho_1, \cdots, \rho_\ell$ that are irreducible, map a fixed maximal torus T of G onto diagonal matrices, and have highest weights $\omega_1, \cdots, \omega_\ell$ satisfying the relations

$$2\langle\omega_i, \alpha_j\rangle / \langle\alpha_j, \alpha_j\rangle = \delta_{ij} \qquad (1 \leq i, j \leq \ell)$$

for a simple system of roots $\{\alpha_1, \cdots, \alpha_\ell\}$ of G with respect to T. The set $\{\rho_1, \cdots, \rho_\ell\}$ is closed under conjugation up to equivalence; that is, the conjugate representation $\bar{\rho}_i$ of ρ_i is equivalent to some ρ_j, so that it can be decomposed into conjugate classes. Each class consists of one self-conjugate representation or two mutually conjugate representations.

In this note we give a general proof of the following theorem.

THEOREM. *The number of even primitive invariants of* G *is equal to the number of conjugate classes of fundamental representations.*

(Professor Armand Borel has pointed out to the author that this theorem can be proved by a method that does not involve KO theory, but still uses Atiyah's result [1].)

The theorem partially illustrates the phenomenon that the number of even primitive invariants is generally larger than the number of odd primitive invariants. (From this point of view, the case $G = SU(2n+1)$ looks rather exceptional.) When no simple factor of G has outer automorphisms, then every ρ_i is self-conjugate, hence every m_i is even.

1. Our basic tool is the rational KO-cohomology, that is, a cohomology theory assigning to each space X (finite CW-complex) the groups $KO^i(X) \otimes Q$ $(i \in Z)$, which we denote by KO_Q [2]. Let

$$\varepsilon: KO^i(X) \to K^i(X) \quad \text{and} \quad \rho: K^i(X) \to KO^i(X) \qquad (i \in Z)$$

denote natural transformations induced by complexification and real restriction of vector bundles respectively [2], [3]. Let g be the generator of K^{-2}(a point) given by the reduced Hopf bundle, let λ and μ be the generators of KO^{-4}(a point) and

Received January 5, 1966 and December 6, 1966.
Work supported in part by NSF grant GP-4069.