ON GENERALIZATIONS OF EULER'S PARTITION THEOREM

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1. INTRODUCTION

Sylvester's memoir on partitions contains an interesting generalization of Euler's partition theorem [12, p. 293 (p. 45 in Collected Works)]. In any partition of n into distinct parts, we may count the total number of sequences of consecutive integers appearing. For example, $31 = 10 + 8 + 7 + 3 + 2 + 1$ consists of three such sequences, namely 10; 8, 7; 3, 2, 1. Sylvester's theorem is as follows.

THEOREM 1. *Let* $A_k(n)$ *denote the number of partitions of* n *into odd parts (repetitions allowed) with exactly* k *distinct parts appearing. Let* $B_k(n)$ *denote the number of partitions of* n *into distinct parts such that exactly* k *sequences of consecutive integers appear in each partition. Then*

$$A_k(n) = B_k(n).$$

For example, let $n = 15$, $k = 3$. Then the partitions enumerated by $A_3(15)$ are

$$11 + 3 + 1, \quad 9 + 5 + 1, \quad 9 + 3 + 1 + 1 + 1, \quad 7 + 5 + 3, \quad 7 + 5 + 1 + 1 + 1,$$

$$7 + 3 + 3 + 1 + 1, \quad 7 + 3 + 1 + 1 + 1 + 1 + 1, \quad 5 + 5 + 3 + 1 + 1, \quad 5 + 3 + 3 + 3 + 1,$$

$$5 + 3 + 3 + 1 + 1 + 1 + 1, \quad 5 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1.$$

Hence $A_3(15) = 11$. The partitions enumerated by $B_3(15)$ are

$$11 + 3 + 1, \quad 10 + 4 + 1, \quad 9 + 5 + 1, \quad 9 + 4 + 2, \quad 8 + 6 + 1, \quad 8 + 5 + 2, \quad 8 + 4 + 2 + 1,$$

$$7 + 5 + 3, \quad 7 + 5 + 2 + 1, \quad 7 + 4 + 3 + 1, \quad 6 + 5 + 3 + 1.$$

Hence $B_3(15) = 11$.

This beautiful theorem was proved arithmetically [12, Section (46)]. F. Franklin has deduced the result for $k = 1$ from a study of the generating functions involved [12, Section (25) (C)]; however, there seems to be no known analytic proof for $k > 1$. In Section 2 of this paper, we prove Sylvester's theorem by means of generating functions.

In Section 3, we give a new generalization of Euler's theorem. Let $\Pi_d(n)$ denote the set of partitions of n into distinct parts. If π is any partition of n, say $b_1 + \cdots + b_s = n$ $(b_i \geq b_{i+1})$, let $g(\pi)$ denote the number of solutions of the inequality $b_i - b_{i+1} \geq 2$ $(i = 1, \cdots, s$; define $b_{s+1} = 0)$. For example, in the partition $18 = 8 + 6 + 2 + 2$, $g(\pi) = 3$.

THEOREM 2. *Let* $C_k(n)$ *denote the number of partitions of* n *with exactly* k *distinct even parts appearing (all other parts being odd), then*

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