INVOLUTIONS FIXING PROJECTIVE SPACES

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The object of this paper is to prove the following result.

THEOREM. *Suppose* (T, M^n) *is a differentiable involution on a closed manifold* M^n $(n > 2r)$, *and its fixed point set is real projective space* $RP(2r)$. *Then* $n = 4r$, *and* (T, M^n) *is cobordant to the involution of* $RP(2r) \times RP(2r)$ *that sends* (x, y) *into* (y, x).

This result was suggested by Conner and Floyd [2, Section 27]. In particular, Conner and Floyd proved that $n = 4r$, and that if $\xi\colon E \to RP(2r)$ denotes the normal bundle of $RP(2r)$ in M^n, then the Stiefel-Whitney class of ξ is $(1 + d)^m$, where both m and the binomial coefficient $\binom{m}{2r}$ are odd, and where d is the nonzero class of $H^1(RP(2r); Z_2)$.

Proof of the theorem. Let $RP(\xi)$ be the total space of the $RP(2r - 1)$-bundle associated with ξ, and let $p\colon RP(\xi) \to RP(2r)$ be the projection. Borel and Hirzebruch [1] have shown that $H^*(RP(\xi); Z_2)$ is the free module over $H^*(RP(2r); Z_2)$, *via* p^*, on the classes $1, c, \cdots, c^{2r-1}$, where c is the characteristic class of the double cover of $RP(\xi)$ by the sphere bundle of ξ. Multiplication in $H^*(RP(\xi); Z_2)$ is given by the formula

$$0 = \sum_0^{2r} c^{2r-i} p^*(w_i(\xi)) = \sum_0^{2r} \binom{m}{i} c^{2r-i}\alpha^i$$

$$= c^{2r} + c^{2r-1}\alpha + \text{terms of higher degree in } \alpha$$

(since m is odd), where $\alpha = p^*(d)$. The Stiefel-Whitney class of $RP(\xi)$ is

$$w = (1+\alpha)^{2r+1}\left\{\sum_0^{2r} \binom{m}{i} (1+c)^{2r-i}\alpha^i\right\}.$$

(See [2, Theorem 23.3].)

By Theorem 28.1 of [2], the antipodal involution on the sphere bundle of ξ bounds a free involution, or equivalently, all of the generalized Stiefel-Whitney numbers $c^i w_\omega[RP(\xi)]$ of $RP(\xi)$ are zero (here w_ω denotes any product $w_{i_1} \cdots w_{i_s}$ of Stiefel-Whitney classes).

Since m and $\binom{m}{2r}$ are odd, $m \geq 2r + 1$. If $m = 2r + 1$, then the bundle ξ and the normal bundle of $RP(2r)$ in $RP(2r) \times RP(2r)$, which is the tangent bundle τ of $RP(2r)$, have the same Stiefel-Whitney class. Thus the bundles $(\xi, RP(2r))$ and

Received March 15, 1966.

The author is indebted to the National Science Foundation for financial support during this work.