A STABILITY CONDITION FOR $y'' + p(x)y = 0$

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In a recent paper, A. C. Lazer [1] showed that if $p(x) > 0$, $p(x) \in C^3(a, \infty)$, $p(x) \to +\infty$ as $x \to \infty$, and

$$(1) \qquad \int_a^\infty |(p^{-1/2}(x))'''|\, dx < +\infty,$$

then all solutions of the equation

$$(2) \qquad y'' + p(x)y = 0$$

satisfy the condition

$$(3) \qquad \lim_{x\to\infty} y(x) = 0.$$

In this note, we establish the same conclusion under weaker hypotheses for the case when p is monotonic.

THEOREM 1. *If* $p'(x) \geq 0$ *for* $a < x < \infty$, $p(x) \in C^3(a, \infty)$,

$$(4) \qquad \lim_{x\to\infty} p(x) = +\infty,$$

and

$$(5) \qquad \int_a^w |(p^{-\alpha}(x))'''|\, dx = o(p^{1-\alpha}(w)) \qquad (w \to \infty)$$

for some α $(0 < \alpha < 1)$, *then every solution of* (2) *satisfies* (3).

Proof. Writing $y^2 + p^{-1}y'^2 = v$ and $p^{-\alpha} = \phi$, we can easily verify the identity

$$(6) \qquad \frac{d}{dx}\left\{p^{1-\alpha}v + \frac{1}{2}\phi'' y^2 - \phi' y' y\right\} = \frac{1}{2}\phi''' y^2 + (1 - 2\alpha)p' p^{-\alpha} y^2.$$

Now we observe that $v' = -p' p^{-2} y^2 \leq 0$; thus $v(x)$ is a positive, nonincreasing function. Therefore

$$(7) \qquad \lim_{x\to\infty} v(x) = s$$

exists. If $s = 0$, then (3) clearly follows. We shall show that the assumption $s > 0$ yields a contradiction. It follows from (4) that all solutions of (2) are oscillatory; hence there exists a sequence $\{x_n\}$ such that $x_n \to \infty$ and

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