NONHOMOGENEOUS DIFFERENTIAL OPERATORS

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1. INTRODUCTION

Let $P = (p_{ij})$ be an n-by-n matrix whose elements are real-valued and continuous on a finite interval [a, b]. Differential operators of the form $LY = Y' + PY$, where Y is an n-by-n matrix, were first seriously studied by Birkhoff and Langer [1], who considered a system consisting of the differential operator $LY = Y' + PY$ and a boundary condition of the form $U(Y) = AY(a) + BY(b) = 0$, where A and B are nonsingular n-by-n matrices.

W. M. Whyburn discussed systems of the form

$$LY = 0, \qquad AY(a) + BY(b) + \int_a^b F(x)Y(x)\,dx = 0,$$

where F is an integrable matrix. He defined an adjoint system whose existence depends on the existence of a solution to $Z' - ZP = F$ that is nonsingular over [a, b] (see [9, pages 53-54]).

R. H. Cole [3] succeeded in defining an adjoint system whose existence depends only on A and B. This system is a slight generalization of the problem discussed by Whyburn. The adjoint, however, is no longer a differential system.

The present paper generalizes the system discussed by Whyburn, but in a different direction. We shall now show that if A, B, C, D are constant matrices and K_1 and K_2 are integrable matrices, then the existence of our adjoint to the system

$$MY = LY + K_2(x)[CY(a) + DY(b)] = 0,$$

$$H(Y) = AY(a) + BY(b) + \int_a^b K_1(x)Y(x)\,dx = 0$$

depends only on A, B, C, and D; moreover, if the adjoint system exists, it has the same form. We shall show that if the system is incompatible, then the nonhomogeneous system $MY = F$, $H(Y) = 0$ has a solution of the form

$$Y(x) = \int_a^b G(x, t)F(t)\,dt,$$

where $G(x, t)$ is a formal solution of the system $MY = 0$, $H(Y) = 0$, for $x \neq t$.

If both the system $MY = 0$, $H(Y) = 0$ and its adjoint are incompatible, the Green's function for the adjoint system is $-G(t, x)$. However, if $MY = 0$, $H(Y) = 0$ is incompatible, it is possible under certain conditions for the adjoint system to be compatible, a situation that does not occur for ordinary differential systems.

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