ON THE FIBRE HOMOTOPY TYPE OF NORMAL BUNDLES

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1. INTRODUCTION

It was proved by Atiyah [1] that the fibre homotopy type of the stable normal sphere bundle of a manifold M is an invariant of the homotopy type of M. Theorem A below (discovered before I learned of Atiyah's proof) gives an elementary proof of this fact, and also applies to nonstable cases. (See [5], for example. One purpose of the present work is to supply item 6 in the bibliography of [5].) The situation considered is a homotopy-commutative diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{f} & M_1 \\ & {\scriptstyle g_0}\searrow \quad \nearrow{\scriptstyle g_1} & \\ & U_1 & \end{array}$$

with f a homotopy equivalence, $g_i\colon M_i \to V$ embeddings (i = 0, 1), and U_1 a closed tubular neighborhood of $g_1(M_1)$. Theorem A implies that the normal sphere bundles of g_0 and g_1 are fibre-homotopically equivalent. Theorem B applies Theorem A to the problem of choosing g_0 (given f and g) so that it will have as many independent normal vector fields as g_1.

The proof of Theorem A in the case $\dim V \geq \dim M + 3$ depends on Lemma 2, due to Milnor, which states that if U_0 is a closed tubular neighborhood of $g_0(M_0)$ inside int U_1, then U_1 - int U_0 is an h-cobordism between the boundaries bU_1 and bU_0. This Lemma is no longer universally true if $\dim V = \dim M + 2$; Theorem C (which is independent from Theorems A and B) exhibits a special case where it is true. An immediate corollary is that *if* $M_0 \times R^k$ *is diffeomorphic to* $M_1 \times R^k$, *then* $M_0 \times S^{k-1}$ *is* h-*cobordant to* $M_1 \times S^{k-1}$. (The interesting case is k = 2.)

All manifolds, immersions, and embeddings are *smooth.*

Throughout the paper, M_0 and M_1 are compact unbounded manifolds of dimension m, and V is a Riemannian manifold of dimension v.

2. FIBRE HOMOTOPY TYPE

If α and β are bundles, then $\alpha \sim \beta$ indicates that α and β are isomorphic, while $\alpha \simeq \beta$ means that α and β have the same *fibre homotopy type.* For this concept, the reader is referred to Dold [3].

Let $f\colon M \to V$ be an immersion. If ν is the normal vector space bundle of f, then $\hat{\nu}$ will denote the normal sphere bundle of f, and conversely.

THEOREM A. *Let* $g_i\colon M_i \to V$ *be embeddings* (i = 0, 1). *Let* $U_1 \subset V$ *be a closed tubular neighborhood of* $g_1(M_1)$ *such that* $g_0(M_0) \subset U_1$. *Let* $f\colon M_0 \to M_1$ *be a homotopy equivalence making a homotopy-commutative diagram*

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