A CLASS OF RELATION TYPES ISOMORPHIC TO THE ORDINALS

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In this paper, we prove that an ordering relation is scattered and homogeneous if and only if for some ordinal ϕ it is isomorphic to the antilexicographically ordered set of all ϕ-termed sequences of integers that are almost always zero. The algebra of all homogeneous scattered types, under ordinal multiplication, turns out to be isomorphic to the ordinals under ordinal addition.

We shall use the notation of [3] with one exception. For the *ordered sum of the relations* G(x) *over* R, we write $\sum_{x,R} G(x)$ rather than $\sum_{R} G(x)$; analogously, for the *ordinal sum of the types* $\gamma(x)$ *over* R, we write $\sum_{x,R} \gamma(x)$ rather than $\sum_{R} \gamma(x)$. An ordering relation R is (one point) *homogeneous*, if for any x, y $\in$ F(R) there exists an automorphism f of R with f(x) = y. Analogously, an order type α is homogeneous if $\alpha = \beta + 1 + \gamma = \beta' + 1 + \gamma'$ implies $\beta = \beta'$ and $\gamma = \gamma'$. We identify the ordinal ϕ with the set of all ordinals less than ϕ. If ϕ has a predecessor, we call the predecessor ϕ - 1. The symbol $\mathfrak{Z}^{\phi}$ stands for the set of all functions on ϕ to the set of integers. If $\phi < \rho$, $M \in \mathfrak{Z}^{\phi}$, $N \in \mathfrak{Z}^{\rho}$, and $M_{\iota} = N_{\iota}$ for every $\iota < \phi$, then we shall refer to N as an *extension of* M. Let ϕ be an ordinal, and let $N \in \mathfrak{Z}^{\phi}$; then Z_N^{ϕ} will denote the relation whose field consists of all elements $M \in \mathfrak{Z}^{\phi}$ such that $M_{\iota} = N_{\iota}$ for almost all (all but finitely many) $\iota < \phi$; the elements of $F(Z_N^{\phi})$ are ordered antilexicographically. The order type of Z_N^{ϕ} is obviously the same for any choice of the sequence N. If for N we choose the function on ϕ with range $\{0\}$, we write Z_0^{ϕ}. Hence, if we use the notation of [2, Chapter VI, Section 3], then

$$\tau(Z_N^{\phi}) = (\omega^* + 1 + \omega)_0^{\phi}.$$

Note that if ϕ is finite, then

$$(\omega^* + 1 + \omega)_0^{\phi} = (\omega^* + \omega)^{\phi},$$

and that

$$(\omega^* + 1 + \omega)_0^{0} = 1.$$

Moreover, for any ordinals ϕ and θ,

$$(\omega^* + 1 + \omega)_0^{\phi} \cdot (\omega^* + 1 + \omega)_0^{\theta} = (\omega^* + 1 + \omega)_0^{\phi+\theta};$$

(see [2, p. 160, (5)]). If there exists a function mapping the ordering relation R isomorphically onto a subrelation of the ordering relation S, we write $R \mathscr{L} S$; if there is no such isomorphism, we write $R \overline{\mathscr{L}} S$. If $\alpha = \tau(R)$ and $\beta = \tau(S)$, we write $\alpha \mathscr{L} \beta$ or

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