ASYMPTOTIC VALUES OF FUNCTIONS HOLOMORPHIC IN THE UNIT DISC

J. E. McMillan

1. INTRODUCTION

This paper was written under the direction of G. R. MacLane, and it is the author's Ph.D. thesis at Rice University. Its main result is an extension of theorems of Bagemihl and Seidel [3, Theorem 3] and MacLane [4, Theorem 11] on the asymptotic values of a function f holomorphic in the unit disc. We say that f *has the asymptotic value* a *at* ζ ($|\zeta| = 1$) *if there exists a Jordan arc that lies in* $\{|z| < 1\}$, *except for the endpoint* ζ, *and on which* f *has the limit* a *at* ζ.

MacLane [4] considered the class $\mathscr{A}$ of nonconstant holomorphic functions having asymptotic values at a dense set of points on $\{|z| = 1\}$. In particular, he proved that *if* $f \in \mathscr{A}$ *and* γ *is an arc of* $\{|z| = 1\}$, *then either* f *has the asymptotic value* ∞ *at a point of* γ *or* f *has point asymptotic values at points of a subset of* γ *of positive Lebesgue measure.* We shall prove a global version of this theorem without the hypothesis $f \in \mathscr{A}$. As corollaries we find that f either has the asymptotic value ∞ or has point asymptotic values on a set of positive measure, and that an f with only finitely many tracts for ∞ must either have only finitely many tracts or have asymptotic values at points of a set of positive measure (for the definition of the concept of a tract, see Section 2). Several related results are also obtained.

2. DEFINITIONS

The following notation will be used throughout this paper. Let $D = \{|z| < 1\}$ and $C = \{|z| = 1\}$. Let f be a function holomorphic in D. For any subset S of the sphere, let

$$A(S) = \{\zeta \in C: \text{there exists } a \in S \text{ such that f has the asymptotic value } a \text{ at } \zeta.\}$$

In particular, we let

$$A = A \text{ (the sphere)}, \qquad A_\infty = A(\{\infty\}), \qquad A^* = A - A_\infty .$$

The Lebesgue measure and exterior Lebesgue measure (in $[0, 2\pi]$) of a subset B of C will be denoted by m(B) and $m_e(B)$. The interior of an arc $\gamma \subset C$ will be denoted by γ^0. If Δ is a plane domain, $\partial\Delta$ will denote the boundary of Δ. The closure of a set S in the plane will be denoted by $\overline{S}$. Also, we write

$$\{|f| > \lambda\} = \{z: |f(z)| > \lambda\}.$$

Let a be a complex number, and suppose that for each $\varepsilon > 0$, $D(\varepsilon)$ is a component of $\{z: |f(z) - a| < \varepsilon\}$; suppose further that $D(\varepsilon_1) \subset D(\varepsilon_2)$ ($\varepsilon_1 < \varepsilon_2$) and

Received May 23, 1964.