A REMARK ON CONTINUED FRACTIONS

H. Davenport

Very little is known about the continued fraction expansion of algebraic numbers other than quadratic irrationals, though Roth's theorem imposes a limitation on the rapidity with which the partial quotients a_n can increase. In particular, it has never been proved that there exists such a number for which the sequence a_n is unbounded, though this is almost certainly true.

The following much easier question was put to me by Dr. Schinzel: given N, however large, do there exist algebraic numbers other than quadratic irrationals which have infinitely many partial quotients greater than N? The following simple result answers this question, but does so without throwing any light on the deeper problems.

Let θ *be any irrational number, and let* P *be any large prime. Then one at least of the numbers*

$$P^2\theta,\ \theta,\ \theta + 1/P,\ \cdots,\ \theta + (P-1)/P$$

has $a_n > P - 2$ *for infinitely many* n.

To prove this, we consider the convergents p_n/q_n to the irrational number $P\theta$. They satisfy the inequality

$$\left|P\theta - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}.$$

If q_n is divisible by P for infinitely many n, we put $q_n = Pq_n'$ and get infinitely many rational approximations p_n/q_n' to $P^2\theta$ satisfying the inequality

$$\left|P^2\theta - \frac{p_n}{q_n'}\right| < \frac{1}{P(q_n')^2}.$$

This implies that infinitely many of the partial quotients in the continued fraction for $P^2\theta$ are greater than P - 2. (This is a consequence of well-known properties of continued fractions; see for example Theorems 184 and 163 in Hardy and Wright's *Introduction to the theory of numbers.*)

In the contrary case, q_n is relatively prime to P for all sufficiently large n. Thus we can determine an integer A_n such that

$$p_n \equiv A_n q_n \pmod{P}, \qquad 0 \le A_n < P.$$

One at least of the integers 0, 1, $\cdots$, P - 1 must occur infinitely often as A_n. If A is such an integer, we can write

$$p_n = Aq_n + Pr_n$$

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