LENGTH AND AREA ESTIMATES FOR ANALYTIC FUNCTIONS

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Suppose that $f(z)$ is analytic for $|z| < 1$, and for $0 < r \leq 1$ let $D(r, f)$ denote the image of $|z| < r$ under $f(z)$. Let $A(r, f)$ denote the area of $D(r, f)$, in the sense that the areas of regions of $D(r, f)$ covered more than once are counted with the appropriate multiplicity. Also, let $L(r, f)$ denote the length of the curve $w = f(re^{i\theta})$ $(0 \leq \theta \leq 2\pi)$. Then, for $r < 1$,

$$A(r, f) \geq \pi r^2 |f'(0)|^2, \tag{1}$$

$$L(r, f) \geq 2\pi r |f'(0)|, \tag{2}$$

and in either estimate equality holds only for the functions $f(z) = a_0 + a_1 z$ [3, p. 155, Problems 10, 11].

We improve (1) and (2) in the following way. If $a(r, f)$ denotes the area of the set $D(r, f)$, then evidently $A(r, f) \geq a(r, f)$. Theorem 1 (with $n = 1$) asserts that $a(r, f) \geq \pi r^2 |f'(0)|^2$. This estimate is precise only for the functions for which (1) is precise.

Let $C(r, f)$ be the outer boundary of $D(r, f)$, and let $\ell(r, f)$ be the length of $C(r, f)$. Here we mean length to be measured as seen; that is, the lengths of multiply covered arcs of $C(r, f)$ are counted only once. Then $L(r, f) \geq \ell(r, f)$, and Theorem 2 (with $n = 1$) asserts that $\ell(r, f) \geq 2\pi r |f'(0)|$. Again, equality occurs only for the functions $f(z) = a_0 + a_1 z$. If $\ell^*(r, f)$ denotes the length of the boundary of $D(r, f)$ (in the above sense) then $L(r, f) \geq \ell^*(r, f) \geq \ell(r, f)$, so that, in particular, our result implies that $\ell^*(r, f) \geq 2\pi r |f'(0)|$.

If $A(1, f)$ exists, then (1) also holds for $r = 1$. Indeed, if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{3}$$

then the formula

$$A(r, f) = \pi \sum_{n=1}^{\infty} n |a_n|^2 r^{2n} \tag{4}$$

is valid even for $r = 1$, in the case where $A(1, f)$ is finite.

If $f(z)$ is continuous for $|z| \leq 1$ and of bounded variation on $|z| = 1$, then (2) is valid for $r = 1$. This follows since (2) holds for all $r < 1$, and with these additional hypotheses $L(r, f) \to L(1, f)$ as $r \to 1$ [5, p. 150, 6.11].

If $f(z)$ has the form (3), then (4) implies that

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