PUSHING A 2-SPHERE INTO ITS COMPLEMENT

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1. INTRODUCTION

We investigate the extent to which a closed set in E^3 can be slightly pushed to one side of a surface. One of the main results (Theorem 2.1) states that if U is a complementary domain of a 2-sphere S (possibly wild) in E^3, then $\overline{U}$ can be slightly shoved into U plus a Cantor set. We show in Theorem 5.1 that for each $\varepsilon > 0$ there exists a Cantor set C on S and an ε-map $f: S \to U + C$ that takes S - C homeomorphically into U. This and other related results are given for surfaces in 3-manifolds as well as for 2-spheres in E^3.

Wilder has given a converse for Theorem 5.1 in which he shows [19] that a compact locally connected continuum S in E^3 is a topological 2-sphere if its first homology is trivial, if S separates E^3, and if for each $\varepsilon > 0$ and each component U of E^3 - S there exists a Cantor set C on S and an ε-map $f: S \to U + C$. He gives a corresponding characterization for 2-manifolds, where instead of supposing that the first homology of S is trivial, he supposes it is finitely generated, and he insists furthermore that C does not locally separate S.

For a 2-sphere S in E^3 we use Int S and Ext S to denote respectively the bounded and the unbounded components of E^3 - S. In case D is a cell, we use Bd D to denote the combinatorial boundary of D, and Int D to denote D - Bd D.

The distance function is denoted by D. In case f, g are maps of a set A into a metric space, we use D(f, g) to denote the least upper bound of D(f(a), g(a)), $a \in A$. We call a map f an *ε-map* if $D(f, I) \leq \varepsilon$, where I is the identity map. A *null sequence* of sets is a sequence of sets whose diameters converge to zero. We use $V(X, \varepsilon)$ to denote the set of all points q whose distance from X is less than ε. We call $V(X, \varepsilon)$ the *ε-neighborhood of* X, but we note that this is only a special sort of neighborhood of X. In general, we call any open set containing X a *neighborhood* of X.

A subset X of E^3 (or of a triangulated manifold M) is called *tame* if there exists a homeomorphism $h: E^3 \to E^3$ (or $M \to M$) such that h(X) is a polyhedron. A closed set which is a topological complex but for which there exists no such homeomorphism is called *wild*. They say that X is *locally tame* at a point p of X if there exists a neighborhood N of p and a homeomorphism h of $\overline{N}$ onto a combinatorial cell that takes $\overline{N} \cdot X$ onto a polyhedron. We say that X is *locally tame* mod K if X is locally tame at each point of X - K. A *finite graph* is the sum of a finite number of arcs (topological segments) such that if two of the segments intersect each other, the intersection is an end point of each.

There are several criteria for determining whether or not a 2-sphere in E^3 is tame [1, 3, 4, 8, 9, 11, 13, 14, 16]. One of the most useful of these [4] says that a 2-sphere is tame if its complement is 1-ULC. Using I^m to denote an m-cell, we say that a set Y is n-ULC if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that each map of Bd I^{n+1} into a δ-subset of Y can be extended to map I^{n+1} into an ε-subset of Y.

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