ON MODULAR FORMS OF LEVELS TWO AND THREE

John Roderick Smart

1. We consider the problem of parametrizing all modular forms of dimension r on certain subgroups of the modular group, namely, on the principal congruence subgroups of levels two and three. As a consequence of the parametrization of the forms, all multiplier systems for the dimension r are determined. We rely on the results of Maak [4] for the determination of all one-dimensional representations for the principal congruence subgroup of level 2. We then use his method to determine the characters on the principal congruence subgroup of level 3.

The parametrization of modular forms for the full modular group has been carried out by Rademacher and Zuckerman [8], [9]. We use arguments similar to those of Section 8 of [9].

2. The homogeneous modular group $\overline{\Gamma}(1)$ consists of all 2-by-2 matrices $V = (a\ b \mid c\ d)$ (written in one line) with rational integral entries and determinant 1. To each element of this group there corresponds a modular substitution

$$Vz = \frac{az + b}{cz + d}.$$

Note that V and $-V = (-a, -b \mid -c, -d)$ correspond to the same substitution. We denote the group of substitutions by $\Gamma(1)$. This group is known to be generated by

$$Sz = z + 1 \qquad \text{and} \qquad Tz = -1/z\,.$$

The principal congruence subgroup of level N, $\overline{\Gamma}(N)$, consists of all those $V \in \overline{\Gamma}(1)$ for which $V \equiv \pm I \pmod{N}$, where $I = (1\ 0 \mid 0\ 1)$, and where we mean element-wise congruence. We shall be interested in cases where $N = 2, 3$.

A fundamental region for the modular group is the set

$$R(1) = \{z = x + iy;\ |z| > 1,\ |x| < 1/2,\ y > 0\}\,.$$

A fundamental region for the group $\Gamma(N)$ is the set

$$\bigcup_{k=1}^{\mu} V_k R(1)\,,$$

where $V_1, \cdots, V_\mu$ are the representatives of the coset decomposition of $\Gamma(1)$ modulo $\Gamma(N)$. Let r be a real number. A modular form of dimension r for the group $\Gamma(N)$ is a meromorphic function $F(z)$ in the upper half-plane which satisfies

$$F(Vz) = v(V)\,(cz + d)^{-r}\,F(z)$$

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