EXTREMAL LENGTH DEFINITIONS FOR THE CONFORMAL CAPACITY OF RINGS IN SPACE

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1. INTRODUCTION

It is well known that many important properties of conformal and quasiconformal mappings in the plane can be deduced from studying what happens to the modulus or, equivalently, to the capacity of rings. Recently Loewner showed in [7] how this method can be extended to higher dimensions by defining a conformal capacity for rings in Euclidean 3-space by means of a Dirichlet integral. The present author has found Loewner's idea very fruitful, and he has used it to establish a number of results on conformal and quasiconformal mappings in space. (See [2], [3] and [4].)

J. Väisälä, in [13] and [14], and B. V. Šabat, in [8] and [9], have investigated quasiconformal mappings in space, using extremal lengths. In doing so, each of them has tacitly introduced a new kind of conformal capacity for a space ring R. Väisälä considered the module of the family of curves in R that join the boundary components of R, while Šabat studied the module of a family of surfaces that separate the boundary components of R.

In the first part of this paper we shall give two extremal length definitions for the conformal capacity of a ring in space. The first of these is essentially equivalent to Väisälä's definition, while the second is a slightly modified version of Šabat's definition. We shall then show that these two extremal length definitions are equivalent to the Dirichlet integral definition due to Loewner.

In the second part of the paper, we use these extremal length definitions to obtain upper and lower bounds for the moduli of some rings in space that have axial symmetry. In particular, we show that the moduli of the Grötzsch and Teichmüller extremal rings in space are greater than or equal to the moduli of the corresponding rings in the plane.

2. NOTATION

We consider sets in the Möbius space, that is, in the finite Euclidean 3-space plus the point at infinity. Points will be designated by capital letters P and Q or by small letters x and y. In the latter case, x_1, x_2, x_3 will denote the coordinates for x, and similarly for y; x will denote the point at infinity if any one of its coordinates x_i is infinite. Points are treated as vectors, and $|P|$ and $|x|$ will denote the norms of P and x, respectively.

Given a point P and sets E and F, we let $\rho(P, E)$ denote the distance between P and E, and $\rho(E, F)$ the distance between E and F. We further let ∂E, $\mathscr{C}E$, and $\overline{E}$ denote the boundary, complement, and closure of E, respectively.

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