DEFINING RELATIONS FOR FULL SEMIGROUPS OF FINITE TRANSFORMATIONS

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1. INTRODUCTION

A transformation f of a set I into itself is said to be *finite* if and only if $f(x) = x$ for all but finitely many elements x of I. Under the operation of composition the set F(I) of all finite transformations of I into itself is a semigroup having the identity map id_I as its identity element. As generators for F(I) we may take all the transpositions (x, y) and replacements (x/y) with $x, y \in I$ and $x \neq y$. Here (x/y) is the transformation that maps y onto x and leaves all the other elements of I fixed, while (x, y) is of course the permutation that interchanges x and y, leaving all the other elements fixed. By an *elementary transformation* we shall mean a transformation that is either a transposition or a replacement.

The purpose of this paper is to give a set of defining relations for F(I), taking the set of all elementary transformations as a generating set. The reason for taking this generating set rather than a smaller irredundant one is that the individual defining relations can then be given in a simple form particularly convenient for applications. This is illustrated in Section 4, where we outline a new proof of a theorem of Galler [1] concerning the relation between cylindric algebras and polyadic algebras.

2. CANONICAL REPRESENTATIONS

We consider a fixed set I consisting of at least three elements. By an *elementary sequence* we mean a finite sequence whose terms are elementary transformations. If $a = \langle a_0, a_1, \cdots, a_{n-1} \rangle$ is an elementary sequence, then we let

$$a^\tau = a_0 a_1 \cdots a_{n-1}.$$

By a *representation* of a member f of F(I) we mean an elementary sequence a with $f = a^\tau$.

Since the set of all elementary transformations obviously generates F(I), every finite transformation f of I has a representation. We shall now single out certain representations of f that will be referred to as *canonical representations*. This concept is motivated by the consideration of the directed graph whose vertices are the elements of I and whose edges are in one-to-one correspondence with the elements of I in such a way that, for each x in I, the corresponding edge has x as its initial vertex and f(x) as its terminal vertex. Let J be the set of all members x of I such that $f^p(x) = x$ for some positive integer p. Clearly f maps J onto itself, and the restriction f' of f to J is a finite permutation. The graph of f' therefore consists of pairwise disjoint cycles, of which all but finitely many are degenerate, consisting of just one vertex.

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