# A GENERALIZATION OF A CONTINUOUS CHOICE FUNCTION THEOREM

J. W. Jaworowski

1. Many applications of the Vietoris Mapping Theorem are known (see, for example, [2] and references in that paper). We give another such application concerning multi-valued functions defined on sets of hyperplanes in Euclidean space. Our theorem generalizes a certain known theorem on continuous choice functions. A special case of the theorem of the present paper was first proved in a different way by A. Kosinski, who subsequently proved a still more general theorem which will be published in [3].

We recall some of the simplest concepts concerning multi-valued functions. A *multi-valued function* $F: X \to Y$, where $X$ and $Y$ are topological spaces, is a function which assigns to each point $x \in X$ a non-empty subset $F(x)$ of $Y$. The set

$$\Gamma(F) = \{(x, y): y \in F(x)\} \subset X \times Y$$

is called the *graph* of $F$.

Let $X$ be a compact space. Then a multi-valued function $F: X \to Y$ is said to be *continuous* if the graph $\Gamma(F)$ is compact. If $Y$ is compact, this condition is equivalent to the upper semi-continuity of $F$, when $F$ is regarded as a single-valued mapping of $X$ into the hyperspace of non-empty compact subsets of $Y$. If $F$ is single-valued, then the above notion of continuity is equivalent to the continuity of a single-valued function in the ordinary sense.

2. We shall use the reduced Čech homology groups $H_k(X)$ of a compact space $X$ with coefficients modulo 2. The space $X$ is called *acyclic* if it is non-empty and if $H_k(X) = 0$ for every $k > 0$. A multi-valued function $F: X \to Y$ is said to be *acyclic* if the sets $F(x)$ are acyclic for every $x \in X$.

VIETORIS MAPPING THEOREM. Let $X$ and $Y$ be compact spaces, and let $f: X \to Y$ be a continuous (single-valued) mapping such that $f^{-1}(y)$ is acyclic for every $y \in Y$. Then $f$ induces an isomorphism $f_*: H_k(X) \approx H_k(Y)$ for every $k = 0, 1, 2, \cdots$ (see [1] and [2], Nr. 3).

3. Let $\mathscr{H}^n_m$ denote the set of all m-dimensional hyperplanes in Euclidean n-dimensional space $E^n$ $(m < n)$ that pass through a fixed point (the origin). For each m-dimensional hyperplane $H \in \mathscr{H}^n_m$, denote by $H^*$ the $(n - m)$-dimensional hyperplane that passes through the origin and is orthogonal to $H$.

THEOREM. *Let* $F$ *be a multi-valued continuous acyclic function which assigns to each hyperplane* $H \in \mathscr{H}^n_m$ *a subset* $F(H)$ *of* $E^n$. *Then there exists a hyperplane* $H_0 \in \mathscr{H}^n_m$ *such that* $F(H_0)$ *intersects* $H_0^*$.

*Proof.* Evidently it is sufficient to prove this theorem for the case when $m = n - 1$. Then the set $\mathscr{H}^n_{n-1}$ is the $(n - 1)$-dimensional projective space $P^{n-1}$. Let $S^{n-1}$ be the unit $(n - 1)$-dimensional sphere in $E^n$ with its center at the origin,

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