BOUNDARY FUNCTIONS FOR FUNCTIONS DEFINED IN A DISK

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1. INTRODUCTION

Let C and D denote respectively the unit circle $|z| = 1$ and the open unit disk $|z| < 1$ in the complex plane. By an *arc at* $\zeta \in C$ we mean a Jordan arc that lies in D except for one end point at ζ. Let $\phi(\zeta)$ and f(z) be real- or complex-valued functions defined on C and D, respectively. (For real-valued functions, we admit $+\infty$ and $-\infty$ as values; for complex-valued functions, we admit ∞ as a value. Instead of the real or complex numbers, we could consider values in more general spaces, but it is doubtful that such extensions would enhance the intrinsic value of our theorems.) We shall say that ϕ is a *boundary function* for f, or that f has a boundary function ϕ, provided that for each $\zeta \in C$ there exists an arc $A(\zeta)$ at ζ such that

$$\lim_{z \to \zeta,\ z \in A(\zeta)} f(z) = \phi(\zeta).$$

When we speak simply of a function f, no restrictions whatever (such as analyticity or continuity, for example) are assumed, unless they are explicitly stated.

In Section 2, we consider the problem of how many different boundary functions ϕ a particular function f can have. Section 3 is concerned with the relation between boundary functions and the Baire classification. In Section 4, finally, we pose a number of problems.

2. THE NUMBER OF BOUNDARY FUNCTIONS POSSESSED BY A FUNCTION

If f is defined in D and if there exist two arcs A and A' at $\zeta \in C$ along which f(z) tends to two distinct limits b and b', respectively, as $z \to \zeta$, we say that ζ is an *ambiguous point* of f. We shall make repeated use of the following fundamental result (see [1, p. 382, Corollary 1]).

THEOREM A. *No function defined in* D *has uncountably many ambiguous points on* C.

We apply this to obtain a theorem on unrestricted functions, and then give examples of various functions having fairly many boundary functions.

THEOREM 1. *Every function* f *in* D *has at most* $2^{\aleph_0}$ *boundary functions.*

Proof. By Theorem A, f has at most $\aleph_0$ ambiguous points. At each ambiguous point, f has at most $2^{\aleph_0}$ asymptotic values. Hence, f has at most $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ boundary functions.

Received April 15, 1961.

This paper was written with partial support from the Air Force Office of Scientific Research, under Contract No. AF 49(638)-633.