MEROMORPHIC FUNCTIONS WITH SMALL CHARACTERISTIC AND NO ASYMPTOTIC VALUES

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1. INTRODUCTION

By the well-known theorem of Fatou, if $f(z)$ is holomorphic and bounded in $|z| < 1$ then $f(z)$ possesses radial limits almost everywhere, that is, $\lim_{r\uparrow 1} f(re^{i\theta})$ exists for almost all θ. This result was extended by Nevanlinna to meromorphic functions of bounded characteristic $T(r)$, for as Nevanlinna showed, the functions meromorphic in $|z| < 1$ with bounded $T(r)$ are exactly those which may be obtained as quotients of bounded holomorphic functions [4, p. 189]. A natural question raised by Lohwater and Piranian [2, p. 16], is this: if the condition of boundedness of $T(r)$ be relaxed to "$T(r)$ doesn't grow faster than so-and-so," can one still conclude that *some* radial limits must exist? Bagemihl, Erdös and Seidel [1, Theorem 7] have given an example of a *holomorphic* function without radial limit for which $T(r) = O((1 - r)^{-8})$. Lohwater and Piranian [2] gave an example of a *meromorphic* function without radial limit for which $T(r) = O(-\log(1 - r))$. See also Noshiro [5, p. 90].

The object of the present paper is to prove (Theorem 5) that there exists a function $F(z)$, *meromorphic* in $|z| < 1$, whose characteristic is dominated by an arbitrarily given increasing unbounded function, such that $F(z)$ has no asymptotic value, finite or infinite, and hence no radial limit. As will be obvious, the method used to prove this result will *not* apply to *holomorphic* functions; a holomorphic function must possess at least one asymptotic value, though not necessarily a radial limit [4, p. 292].

It is of interest to note one result for holomorphic functions which follows easily from theorems of Zygmund [6, pp. 90-91]. A slight reformulation of these results of Zygmund may be stated as

THEOREM 1. *Let*

$$F(z) = \sum_{k=1}^{\infty} c_k z^{n_k},$$

where

$$(1) \qquad n_{k+1}/n_k > q > 1, \quad \limsup |c_k|^{1/n_k} = 1, \quad \lim c_k = 0, \quad \sum_{k=1}^{\infty} |c_k|^2 = \infty .$$

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