HOMOTOPICALLY HOMOGENEOUS POLYHEDRA

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1. INTRODUCTION

If X is a space, let

$$\hat{X} = \{(x_1, x_2) \mid x_1 \neq x_2\} \subset X \times X,$$

and define $p: \hat{X} \to X$ by $p(x_1, x_2) = x_1$. We say that X is *homotopically homogeneous* (abbreviated h.h.) if $(\hat{X}, X, p)$ is a Hurewicz fiber space, that is, if $(\hat{X}, X, p)$ has the covering homotopy property for maps of any topological space. It follows immediately that if y, z are points of a path-wise connected h.h. space, then X - y and X - z have the same homotopy type.

A homogeneous polyhedron is clearly a manifold, and h.h. polyhedra turn out to be a kind of homotopy manifold. In particular, an h.h. polyhedron is a Kosiński r-polyhedron [5], and also a homotopy manifold as defined by Griffiths [4]. Thus h.h. polyhedra of dimensions 1, 2, 3 are manifolds, and 4-dimensional h.h. polyhedra are manifolds if the Poincaré Conjecture is true. No example is known of an h.h. polyhedron which is not a manifold.

Section 2 gives some examples. Manifolds, groups, and loop spaces are h.h., and closed cells are not. In Section 3 a class of *locally conical* spaces is considered, so that results are a little more general than for polyhedra. Some lemmas on covering homotopies for locally conical h.h. spaces are proved. The results of Section 3 are applied in Section 4 to show that locally conical h.h. spaces are Kosiński r-spaces, homotopy manifolds, and (hence) homology manifolds. Section 5 is devoted to the consideration of locally conical homology manifolds. A different proof is given of the theorem of Kwun and Raymond [6] that 3-dimensional locally conical generalized manifolds are locally euclidean. Combining these results with those of Section 4 shows that, modulo the Poincaré Conjecture, 4-dimensional h.h. polyhedra are manifolds.

2. EXAMPLES

Example 1. Any n-manifold (separable metric locally euclidean space) is an h.h. space.

We shall show that if M is an n-manifold, then $(\hat{M}, M, p)$ is a locally trivial fiber space. It will then follow from [2] that $(\hat{M}, M, p)$ is a Hurewicz fiber space. Let B be the open unit ball in E^n, and let C be the ball concentric with B and with radius 1/2. Given a point m in M, there exists a homeomorphism h of B into M sending the origin to m. Let h(B) = P and h(C) = Q. We need to define a homeomorphism

$$f: Q \times (M - m) \to p^{-1}(Q)$$

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