ON THE STRUCTURE OF SEMIGROUPS WITH IDENTITY ON A NONCOMPACT MANIFOLD

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This paper deals with a problem concerning the structure of a semigroup with identity whose space is a noncompact n-dimensional manifold ($n \geq 2$), and which has an $(n - 1)$-dimensional compact connected group containing the identity. This problem is analogous to the one considered by Mostert and Shields in [7].

Definition 1. A *thread* is a semigroup whose space is homeomorphic to an open interval.

Definition 2. An M-*thread* is a semigroup with identity whose space is homeomorphic to a half-open interval and such that the endpoint acts as a zero (see [6]).

The term "isomorphism" will be used to denote a function which is simultaneously an algebraic isomorphism and a homeomorphism. For further definitions and background material, the reader is referred to [7] and [10].

THEOREM. *Let* M *be an* n-*dimensional noncompact manifold* ($n \geq 2$) *which is a semigroup with identity,* 1, *and assume that* M *is not a group. Suppose there exists a compact connected* $(n - 1)$-*dimensional group* G *containing* 1 *and contained in* M. *Then there exists either a thread with identity, or else an* M-*thread,* T, *such that* $M = TG$ *and* $tg = gt$ *whenever* $t \in T$ *and* $g \in G$.

The proof of the theorem is divided into two main cases. If G is allowed to act on M by right multiplication, then the orbit space with respect to this action is either an open interval or a half-open interval. The cases in the proof correspond to the two kinds of orbit space. In the following twenty-two lemmas that comprise the proof, the hypotheses of the theorem are assumed to hold throughout.

LEMMA 1. *Define* $\theta: M \times G \to M$ *by* $\theta(x, g) = xg$ *for* $x \in M$, $g \in G$. *Then* G *is a Lie group acting effectively on* M. *If* S *denotes the orbit space with respect to* G, *then* S *is homeomorphic to a half-open interval or an open interval.*

Proof. By assumption, M is an n-dimensional manifold and a semigroup with identity. By Mostert and Shields [8], H(1), the maximal subgroup of M containing 1, is an open subset of M and is a Lie group. G, being a closed subgroup of H(1), is therefore also a Lie group.

From the definition of θ given above it follows quite easily, since multiplication in M is continuous and associative, that G acts as a transformation group on M. To see that G acts effectively, it is sufficient to note that G acts by right multiplication, and that the identity for G is also an element of M.

This proves that G is a compact connected Lie group acting effectively on an n-dimensional manifold. To prove that the orbit space is as stated, it suffices to establish the existence of at least one $(n - 1)$-dimensional orbit. This follows

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