SUBRINGS OF SIMPLE ALGEBRAS

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This paper is essentially an appendix to the work [1] of R. A. Beaumont and the author. Its purpose is to clarify the concept introduced there of the smallest field of definition for a subring of a simple rational algebra. However, the main results can be formulated for subrings of quite general algebras, and the proofs do not depend on the developments in [1]. We are indebted to the referee for this observation and for substantial simplification of the paper generally.

Let Λ be an integral domain, and suppose that Q is the quotient field of Λ. Throughout the paper, S is to be a finite-dimensional Q-algebra containing the subring A such that A is a Λ-module and $QA = S$. Let C be the center of A. A field F is called a *field of definition* of A if $\Lambda \subset F \subset C$ and there exist an F-basis $a_1, \cdots, a_k$ of S in A and a nonzero element $\lambda \in \Lambda$ such that

$$\lambda A \subset (A \cap F)a_1 + \cdots + (A \cap F)a_k. \tag{1}$$

It is routine to show that this property does not depend on the choice of $a_1, \cdots, a_k$. In case Λ is the ring of integers, the last condition is equivalent to this, that the group $(A \cap F)a_1 + \cdots + (A \cap F)a_k$ is of finite index in A. If also A contains the identity 1 of S, then $F \subset C$ is a field of definition of A if and only if A is a finitely generated $A \cap F$-module.

For any Λ-submodule B of a Q-space T, define

$$QE(B) = \{h \in \mathrm{Hom}_Q(T, T) \mid \lambda h(B) \subset B \text{ for some } \lambda \neq 0 \text{ in } \Lambda\}. \tag{2}$$

If B satisfies $QB = T$ and $h \in \mathrm{Hom}_\Lambda(B, B)$, then h can be extended to T by defining $h(t) = \lambda^{-1}h(\lambda t)$, where $\lambda \neq 0$ in Λ is such that $\lambda t \in B$. Thus, $\mathrm{Hom}_\Lambda(B, B)$ can be identified with $E(B) = \{h \in \mathrm{Hom}_Q(T, T) \mid h(B) \subset B\}$. Consequently, by (2), $Q \otimes_\Lambda \mathrm{Hom}_\Lambda(B, B)$ can be identified with $Q(E(B)) = QE(B)$. In particular, $QE(B)$ does not depend on the manner in which B is imbedded in T.

If F is any field between Q and C, then $\mathrm{Hom}_F(S, S)$ can be identified with the subring of $\mathrm{Hom}_Q(S, S)$ consisting of all Q-endomorphisms which commute with multiplication by elements of F. Henceforth this identification will be made.

LEMMA 1. *The field* F *is a field of definition of* A *if and only if* $\Lambda \subset F \subset C$ *and* $\mathrm{Hom}_F(S, S) \subset QE(A)$.

Proof. Let $a_1, \cdots, a_k$ be an F-basis of S in A. Put

$$B = (A \cap F)a_1 + \cdots + (A \cap F)a_k.$$

Assume that F is a field of definition of A, and let $\lambda \neq 0$ in Λ be such that $\lambda A \subset B \subset A$. Let $h \in \mathrm{Hom}_F(S, S)$. Since $QA = S$, there exists $\mu \neq 0$ in Λ such that $\mu h(a_i) \in A$ for $i = 1, \cdots, k$. Then

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