REPRESENTATIONS OF A CLASS OF INFINITE GROUPS

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1. INTRODUCTION

The results of this paper are contained in two theorems.

THEOREM 1. *A free group with a countable number of generators can by faithfully represented by a group of 2-by-2 unimodular matrices with rational integral entries (that is, by a subgroup of the modular group).*

THEOREM 1. *Let* G *be a group with a countable number of generators* t_j $(j = 1, 2, \cdots)$ *and a finite number of generating relations*

$$t_1^{q_1} = t_2^{q_2} = \cdots = t_n^{q_n} = 1 \qquad (q_j = \text{integer} \geq 2)\,. \tag{1}$$

Let $\mathscr{F}_n$ *be the algebraic number field obtained by adjoining the quantities*

$$\lambda_j = 2\cos \pi/q_j \qquad (j = 1, 2, \cdots, n) \tag{2}$$

to the rational field. Then G *can be faithfully represented by a group of* 2-*by*-2 *unimodular matrices whose entries are integers in* $\mathscr{F}_n$.

The proof will show that in each case infinitely many inequivalent representations are possible.

It will also be clear from the construction that Theorem 2 is valid for a group G whose generators fall into two classes A and B; those in A have power relations of type (1), those in B are free. Either A or B or both may be finite or denumerably infinite; either may be empty.

An application of these results to the theory of discontinuous groups of linear transformations of the plane is made in Section 5.

Such representations are useful in the construction of subgroup topologies for groups of the above types. (See [3].) The construction will be carried out in a future publication.

2. THE ISOMETRIC CIRCLE

Our main tool will be the isometric circle of L. R. Ford [2, p. 23 ff]. Let z be a complex variable. Given a linear transformation of the plane $z' = (az + b)/(cz + d)$, with $ad - bc = 1$ and $c \neq 0$, we define the circle

$$I(T):\ |cz + d| = 1$$

and call it the isometric circle of T. If T has an isometric circle (that is, if $c \neq 0$), then so has T^{-1}. The isometric circle I(T) together with its interior will be called the isometric disk of T, and we shall denote it by K(T).

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