A NOTE ON THE LAGUERRE POLYNOMIALS

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1. Burchnall [2] employed the operational formula

$$(1)\qquad (D - 2x)^n = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} H_{n-r}(x)\, D^r ,$$

where $D = d/dx$ and

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2} ,$$

to prove the identity [3, p. 31]

$$(2)\qquad H_{m+n}(x) = \sum_{r=0}^{\min(m,n)} (-2)^r \binom{m}{r} \binom{n}{r} r!\, H_{m-r}(x)\, H_{n-r}(x) .$$

Put

$$(3)\qquad L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x D^n (x^{\alpha+n} e^{-x}) ,$$

the Laguerre polynomial of degree n. Corresponding to (1) we shall show that

$$(4)\qquad \prod_{j=1}^{n} (xD - x + \alpha + j) = n! \sum_{r=0}^{n} \frac{1}{r!} x^r L_{n-r}^{(\alpha+r)}(x)\, D^r .$$

Note that the linear operators on the left of (4) commute.

To prove (4) we show first that

$$(5)\qquad \Omega_n y = x^{-\alpha} e^x D^n (x^{\alpha+n} e^{-x} y) ,$$

where y is an arbitrary (differentiable) function of x and

$$\Omega_n = \prod_{j=1}^{n} (xD - x + \alpha + j), \qquad \Omega_0 = 1 .$$

Clearly (5) holds for $n = 0$. Now

$$\begin{aligned} x^{-\alpha} e^x D^{n+1} (x^{\alpha+n+1} e^{-x} y) &= x^{-\alpha} e^x D^{n+1} (x \cdot x^{\alpha+n} e^{-x} y) \\ &= x^{-\alpha} e^x \{ x D^{n+1} (x^{\alpha+n} e^{-x} y) + (n+1) D^n (x^{\alpha+n} e^{-x} y) \} \\ &= x^{-\alpha} e^x \{ x D (x^{\alpha} e^{-x} \Omega_n y) + (n+1) x^{-\alpha} e^x \Omega_n y \} \end{aligned}$$

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