A NOTE ON MATRIX COMMUTATORS

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The problem of representing a given matrix as a matrix commutator has received attention from several authors. (See for example [1], [5], [6], or [7].) Motivated by a recent paper of I. N. Herstein [2], this note provides necessary and sufficient conditions for representing a given nonsingular matrix as a multiplicative commutator $ABA^{-1}B^{-1}$ such that the additional condition $A(AB - BA) = (AB - BA)A$ is satisfied.

LEMMA. *Let* D *be a nonsingular* n-*by*-n *matrix over a field* F *of characteristic zero or prime* $p > n$. *Then there exist nonsingular matrices* A *and* B *over* F *such that* $D = ABA^{-1}B^{-1}$ *and* $A(AB - BA) = (AB - BA)A$ *if and only if* $D - I$ *is nilpotent.*

Necessity. The necessity of this condition is a restatement of a theorem proved by C. R. Putnam and A. Wintner [3] for fields of characteristic zero and by I. N. Herstein [2] for fields of prime characteristic $p > n$.

Sufficiency. Let $D - I$ be nilpotent. Since $D - I$ is similar to a direct sum of matrices, each nilpotent of index equal to its order, it is clearly sufficient to prove the result for the n-by-n matrix $D - I$ that is nilpotent of index n. Furthermore, without loss of generality, let $D = (d_{ij})$, where $d_{ij} = \delta_{ij} + \delta_{i,j-1}$, be in classical canonical form.

Let $A = (a_{ij})$, where $a_{ij} = \binom{j}{i}$ for $i \leq j$ and $a_{ij} = 0$ otherwise. By matrix multiplication and by the addition properties of binomial coefficients, it is easily shown that $A(2I - D) = (a_{i-1,j-1})$ and that $DA(2I - D) = A$. Hence $D^{-1}A = A(2I - D)$ and $A(D - I) = (I - D^{-1})A$. By the restriction on the characteristic of the field, it is evident that none of the elements $a_{k-1,k} = k$ of the first superdiagonal of either A or $D^{-1}A$ are zero. Hence, both $A - I$ and $D^{-1}A - I$ are nilpotent of index n, and this implies that A and $D^{-1}A$ are similar. Thus, there is a nonsingular matrix B such that $D^{-1}A = BAB^{-1}$, and since A is nonsingular, $D = ABA^{-1}B^{-1}$. Finally,

$$A(AB - BA) = A(ABA^{-1}B^{-1} - I)BA = A(D - I)BA = (I - D^{-1})ABA$$

$$= (I - BAB^{-1}A^{-1})ABA = (AB - BA)A.$$

The preceding proof suggests the following theorem.

THEOREM. *Let* D *be a nonsingular* n-*by*-n *matrix over a field* F *other than the field of two elements. Then there exist nonsingular matrices* A *and* B *over* F *such that* $D = ABA^{-1}B^{-1}$ *and* $A(AB - BA) = (AB - BA)A$ *if and only if* $D - I$ *is similar to* $I - D^{-1}$.

Necessity. Let $D = ABA^{-1}B^{-1}$ and $A(AB - BA) = (AB - BA)A$. Then

$$A(D - I) = A(ABA^{-1}B^{-1} - I) = A(AB - BA)A^{-1}B^{-1} = (AB - BA)AA^{-1}B^{-1}$$

$$= (AB - BA)B^{-1}A^{-1}A = (I - BAB^{-1}A^{-1})A = (I - D^{-1})A.$$

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