BOUNDED J-FRACTIONS AND UNIVALENCE

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INTRODUCTION

Some attention has recently been given, by Scott, Thale, and Perron, to the problem of finding the domain of univalence of certain well-known classes of continued fractions. In particular, Thale [4] obtained a circular domain of univalence for the class of bounded S-fractions, and one for the class of bounded J-fractions. Perron [3] has established the fact that the result of Thale on bounded S-fractions is sharp. In this paper it is shown that Thale's circular domain of univalence for the class of bounded J-fractions cannot be enlarged. Moreover, some properties related to univalence are obtained for the latter class of continued fractions.

1. THE RADIUS OF UNIVALENCE

Let $M \geq 0$ and $N > 0$ be real numbers. Consider the class $J(M, N)$ of functions of the form

$$\frac{1}{z+b_1} - \frac{a_1^2}{z+b_2} - \cdots - \frac{a_n^2}{z+b_{n+1}} - \cdots, \tag{1.1}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences of complex numbers such that

$$|a_n| \leq N/3, \qquad |b_n| \leq M/3 \qquad (n = 1, 2, \cdots). \tag{1.2}$$

It is known [5, p. 112] that every function in the class $J(M, N)$ is regular for $|z| > (2N + M)/3$.

Thale [4] has shown that each function of $J(M, N)$ is univalent for

$$|z| > (3\sqrt{2}N + 2M)/6.$$

The function

$$\frac{1}{z+} \frac{N^2/9}{z - M/3 -} \frac{N^2/9}{z - M/3 -} \cdots \frac{N^2/9}{z - M/3 -} \cdots = \frac{6}{9z - M - \sqrt{(3z-M)^2 - 4N^2}},$$

whose derivative is zero at $z = (3\sqrt{2}N + 2M)/6$, shows that *there is no larger circular domain of univalence for the class* $J(M, N)$. By an equivalence transformation, the function $e^{i\theta} f(e^{i\theta} z)$ for fixed θ $(0 \leq \theta < 2\pi)$ is in the class $J(M, N)$ whenever $f(z)$ is in the class. Thus there does not exist a domain of univalence for the class $J(M, N)$ which properly contains the disk $|z| > (3\sqrt{2}N + 2M)/6$.

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