ON SOME METRIC PROPERTIES OF POLYNOMIALS WITH REAL ZEROS

Ch. Pommerenke

1. Let $f(z) = \Pi_{\nu=1}^{n}(z - x_\nu)$ be a polynomial of degree n with real zeros x_ν, and let E be the set $|f(z)| \le 1$. The real axis is denoted by X. A circle (semicircle) that has a segment of X as diameter will be called an orthogonal circle (semicircle).

LEMMA. *Let* L *be an orthogonal semicircle over the real points* a_1 *and* a_2. *If* $z_0 \in E \cap L$ *and* $\Im z_0 > 0$, *then either the arc* $a_1 z_0$ *or the arc* $z_0 a_2$ *of* L *is contained in* E.

Proof. (I owe the idea of this argument to [1, p. 139].) If $x_1 = \cdots = x_n = 0$, the lemma is trivially true. Therefore we can assume that not all of these equations hold. We may take $a_1 = -\rho$, $a_2 = \rho$. Then we have L: $z = \rho e^{i\theta}$ $(0 \le \theta \le \pi)$. Consider the function

$$G(\theta) = \log|f(\rho e^{i\theta})| = \frac{1}{2}\sum_{\nu=1}^{n} \log(\rho^2 - 2\rho x_\nu \cos\theta + x_\nu^2)$$

for $0 < \theta < \pi$. Its derivative is

$$G'(\theta) = \rho(\sin\theta)H(\theta), \qquad \text{where } H(\theta) = \sum_{\nu=1}^{n} \frac{x_\nu}{\rho^2 - 2\rho x_\nu \cos\theta + x_\nu^2}.$$

Differentiating $H(\theta)$, we obtain

$$H'(\theta) = \sum_{\nu=1}^{n} \frac{-\rho x_\nu^2 \sin\theta}{(\rho^2 - 2\rho x_\nu \cos\theta + x_\nu^2)^2} < 0$$

for $0 < \theta < \pi$. Hence $G'(\theta) = \rho(\sin\theta)H(\theta)$ has at most one zero σ in $0 < \theta < \pi$. The relation

$$G''(\sigma) = \rho(\cos\sigma)H(\sigma) + \rho(\sin\sigma)H'(\sigma) = \rho(\sin\sigma)H'(\sigma) < 0$$

shows that $G(\theta)$ has a maximum in σ. Therefore the function $G(\theta)$ does not assume a minimum in $0 < \theta < \pi$. Put $z_0 = \rho e^{i\theta_0}$. We have to consider three cases:

1. $G(0) > 0$. Since the function $G(\theta)$ has no minimum in $0 < \theta < \pi$, the inequality $G(\theta_0) \le 0$ implies that $G(\theta) \le 0$ for $\theta_0 \le \theta \le \pi$. Therefore the arc $a_1 z_0$ of L belongs to E.

2. $G(\pi) > 0$. We can show in a similar way that the arc $z_0 a_2$ of L is contained in E.

3. $G(0) \le 0$, $G(\pi) \le 0$. If neither $a_1 z_0$ nor $z_0 a_2$ were contained in E, there would be two values θ_1 and θ_2 such that $0 < \theta_1 < \theta_0 < \theta_2 < \pi$ and $G(\theta_1) > 0$,

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