THE POSITIVITY SETS OF THE SOLUTIONS OF A TRANSPORT EQUATION

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1. Let

$$\text{(1)} \qquad Df(x, t) = 0, \ f(x, 0) = g(x) \geq 0 \qquad (x, t \geq 0)$$

be an initial-value problem whose operator D and initial function $g(x)$ are such that the following properties hold:

(i) there exists a unique continuous solution $f(x, t)$ valid for $x, t \geq 0$,

(ii) $f(x, t)$ is analytic in t for each x,

(iii) $f(x, t) \geq 0 \ (> 0)$ if $g(x) \geq 0 \ (> 0)$.

Let $P = \{x \mid g(x) > 0\}$ and $Z = \{x \mid g(x) = 0\}$, and define

$$Z_n = \left\{ x \,\middle|\, \frac{\partial^k f(x, 0)}{\partial t^k} = 0 \ (k = 0, 1, \cdots, n-1), \ \frac{\partial^n f(x, 0)}{\partial t^n} > 0 \right\} \qquad (n = 1, 2, \cdots),$$

$$Z_\omega = Z - \bigcup_1^\infty Z_n.$$

Z_n is called the n-th positivity set, and Z_ω is called the residual set; the totality of these gives some information about the behaviour of $f(x, t)$, especially for small t. For example, $f(x, t) > 0$ for $t > 0$ if and only if $x \in Z \cup P - Z_\omega$; $f(x, t) = 0$ for all t if and only if $x \in Z_\omega$; and over Z_n, $f(x, t) = O(t^n)$ for small t.

In this note there will be considered an example of a nonlinear integrodifferential operator D for which the sets Z_n can be completely described in terms of Z and P alone.

2. The equation

$$\text{(2)} \qquad \frac{\partial f(x, t)}{\partial t} = \frac{1}{2}\int_0^x f(y, t) f(x - y, t)\, \phi(y, x - y)\, dy - f(x, t) \int_0^\infty f(y, t)\, \phi(x, y)\, dy$$

has been considered, as a special case, in [1]. It satisfies the above conditions (i), (ii), and (iii) under the following hypotheses:

(H_1) $f(x, 0)$ is a continuous, nonnegative, integrable and uniformly bounded function for $x \geq 0$, and

(H_2) $\phi(x, y) = \phi(y, x)$ is a continuous, nonnegative and uniformly bounded function for $x, y \geq 0$.

Received September 10, 1958.