RIGHT-ORDERED GROUPS

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1. INTRODUCTION

In this note, "order" will always mean linear order. A group is *right-ordered* (notation: it is an ro-group) if it is an ordered set and if multiplication on the right preserves this order ($a < b$ inplies $ac < bc$). In the process of investigating the group $\mathscr{A}$ of order-preserving automorphisms of an ordered group ([3] and [4]), it became apparent that $\mathscr{A}$ could always be right-ordered. In Section 5 it is shown that every group of order-preserving permutations of an ordered set can be right-ordered. Also, every ro-group G is o-isomorphic to a subgroup of the ro-group of all o-permutations of the set G. In Section 4 (Theorem 4.1), we prove that the following two properties of an ro-group G are equivalent: (a) for each pair of positive elements a, b in G, there exists a positive integer n such that $(ab)^n > ba$; (b) if C and C' are convex subgroups of G, and C' covers C, then C is normal in C' and there exists an order-preserving isomorphism of C'/C into the additive group of real numbers. In Section 2 it is shown (Theorem 2.1) that a right-ordering of G is an ordering if and only if $a < b$ implies $b^{-1} < a^{-1}$ for all a, b in G. We also derive four properties, each of which is a necessary and sufficient condition for the right-ordering of a group. In Section 3, some well-known properties of ordered groups are shown to hold for ro-groups.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR A GROUP TO ADMIT A RIGHT-ORDERING

Let G be a group with identity e. Then G is an ro-*group* provided

(1) $(G, <)$ is an ordered set;

(2) if $a < b$, then $ac < bc$ for all a, b, c in G.

LEMMA 1.1. *A group* G *admits a right-order if and only if there exists a subsemigroup* P *of* G *that satisfies*

(*) $\quad e \notin P$; *if* $g \neq e$ *and* $g \in G$, *then* $g \in P$ *or* $g^{-1} \in P$.

The proof is entirely similar to the one for o-groups. For if G is right-ordered, let $P = \{g \in G: g > e\}$, and if P is a subsemigroup of G that satisfies (*), then define $a < b$ if $ba^{-1} \in P$. P is the *semigroup of positive elements of* G.

COROLLARY I. *If* G *is an* ro-*group, then* G *is torsion-free.*

For consider $e \neq g \in G$. If $e < g$, then $e < g < g^2 < \cdots$, and if $g < e$, then $\cdots < g^2 < g < e$.

COROLLARY II. *If* G *is abelian, then every right-ordering of* G *is an ordering. Every abelian subgroup of an* ro-*group is an* o-*group.*

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