SOME REMARKS ON FUNDAMENTAL SOLUTIONS OF PARABOLIC DIFFERENTIAL EQUATIONS OF SECOND ORDER

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1. INTRODUCTION

Let E^n be the real n-dimensional Euclidean space of points $x = (x_1, \cdots, x_n)$, and $D \subset E^n$ an open simply connected domain. Let

$$(1.1) \qquad L(u) = \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right) - V(x)u \qquad (a_{ik} = a_{ki},\ V \geq 0)$$

be a uniformly elliptic operator in D with coefficients depending on x. Let I_T be the interval $0 < t < T$, and Δ_T the product $D \times I_T$. A fundamental solution of the parabolic equation

$$(1.2) \qquad \Lambda \equiv L(u) - \frac{\partial u}{\partial t} = 0$$

in $\Delta = \Delta_\infty$ may be defined as a function $\Gamma(x, \xi, t)$ which as function of (x, t) satisfies (1.2) in Δ, and in addition has the following property: for each function $h(x)$ which is continuous in the closure $\overline{D}$ of D and for each (proper or improper) subdomain D_1 of D, the limit relation

$$(1.3) \qquad \lim_{t \to 0} \int_{D_1} h(\xi)\, \Gamma(x, \xi, t)\, d\xi = \begin{cases} h(x) & \text{for } x \text{ interior to } D_1, \\ 0 & \text{for } x \text{ interior to } D - D_1 \end{cases}$$

holds [19], [6], [7].

It is known that if D is bounded and has a smooth enough boundary $\dot{D}$, then such a fundamental solution may be constructed as follows: let $\{u_1(x), u(x), \cdots\}$ be a full orthonormal set of eigenfunctions, and $\{-\lambda_1, -\lambda_2, \cdots\}$ the set of corresponding eigenvalues of the elliptic eigenvalue problem

$$(1.4) \qquad L(u) - \lambda u = 0 \quad \text{in } D,$$

$$(1.5) \qquad u = 0 \quad \text{on } \dot{D};$$

then

$$(1.6) \qquad G(x, \xi, t) = \sum_{k=1}^{\infty} u_k(x)\, u_k(\xi)\, e^{-\lambda_k t}$$

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