# SEQUENCES OF LINEAR FRACTIONAL TRANSFORMATIONS

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A point set E in the extended z-plane will be called an SD (set of divergence) provided there exists a sequence of transformations

$$T_n(z) = (a_n z + b_n)/(c_n z + d_n)$$

that diverges at each z in E and converges at each z in the complement of E. In the present paper, we give a topological characterization of the SD's that lie on a straight line.

We also characterize the denumerable SD's. But for this purpose, topological ideas are not sufficient (see [1, p. 133]), and we introduce a geometric analogue to the concept of a limit point.

## 1. SETS OF DIVERGENCE ON A STRAIGHT LINE

THEOREM 1. *If a set* E *lies on a straight line, it is an* SD *if and only if it is of type* $G_{\delta\sigma}$.

The necessity of the condition follows immediately from the fact that the transformations $T_n$ are continuous, in the extended plane.

In proving the sufficiency, we may assume, without loss of generality, that the set E lies on the extended real axis. If E coincides with the extended real axis, it is of type $G_\delta$; this case is covered by Theorem 3 of [1]. In the other case, we may assume that the point $z = \infty$ does not belong to E, so that E can be represented in the form

$$E = \bigcup_{j=1}^{\infty} E_j, \qquad E_j = \bigcap_{k=1}^{\infty} E_{jk},$$

where for each j the family $\{E_{jk}\}_{k=1}^{\infty}$ constitutes a decreasing sequence of open sets on the segment $(-j/2, j/2)$ of the real axis. (Even if E is empty, we may assume that none of the sets $E_{jk}$ is empty.) For each index pair (j, k), we denote by $\{E_{jkp}\}$ the finite or denumerable family of components $(a_{jkp}, b_{jkp})$ of $E_{jk}$. With each interval $E_{jkp}$, we associate a domain $B_{jkp}$ bounded by $E_{jkp}$ and by arcs of the two parabolas

$$y = (jkp)^{-1}(x - a_{jkp})^2, \qquad y = (jkp)^{-1}(x - b_{jkp})^2. \tag{1}$$

We construct a denumerable set of circular disks $D_{jkpq}$ (see Figure 1) with centers $z_{jkpq} = x_{jkpq} + i y_{jkpq}$, subject to the following three requirements:

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