REMARKS ON A PAPER BY A. FRIEDMAN

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In this note we shall give a slight generalization of Theorem 2 in the preceding paper by Friedman, and we shall remark on additional problems in the same direction. We adhere to Friedman's notation.

In the following generalization of Friedman's Theorem 2, we assume that n and p are fixed positive integers with $n > 2p$, that ϕ is a fixed real number, and that $[(\alpha, \beta), n, R, \phi]$ denotes the closed finite region bounded by a certain oriented polygon centered at (α, β).

THEOREM 2'. *Let* $u(x, y)$ *be a real-valued function, continuous in the domain* D. *If there exist real-valued functions* $A_k(x, y)$ $(0 \leq k \leq p - 1)$, *continuous in* D, *and with* $A_0 \equiv 1$, *such that*

$$\text{(i)} \qquad \frac{1}{sR^2} \iint_{[(\alpha,\beta),n,R,\phi]} u(x, y)\,dx\,dy = \sum_{k=0}^{p-1} A_k(\alpha, \beta) R^{2k}$$

holds for all $[(\alpha, \beta), n, R, \phi]$ *in* D, *then* $u(x, y)$ *is a* p-*harmonic polynomial of degree at most* pn, *and its derivative in the* ϕ-*direction vanishes identically.*

Proof. As Friedman remarks, it is sufficient to consider the case $\phi = 0$, for the general case can then be obtained by a rotation. We note that Friedman's formula (2.1) can be modified to yield the following result:

$$\text{(ii)} \qquad \sum_{k=1}^{n} \left[\left(\cos \frac{2\pi k}{n}\right) \frac{\partial}{\partial x} + \left(\sin \frac{2\pi k}{n}\right) \frac{\partial}{\partial y} \right]^t u(\alpha, \beta) = \begin{cases} \dfrac{n}{4m} \dbinom{2m}{m} \triangle^{2m} u(\alpha, b) & (0 < t = 2m < n), \\ 0 & (0 < t = 2m - 1 < n). \end{cases}$$

First we assume that $u(x, y)$ has continuous partial derivatives of the first 2p orders in D. If we develop $u(x, y)$ in a finite Taylor expansion about an arbitrary point (α, β) in D,

$$u(x, y) = \sum_{k=0}^{2p} \frac{p^k}{k!} \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right)^k u(\alpha, \beta) + o(p^{2p}),$$

and make use of (ii), we obtain

$$\text{(iii)} \qquad \frac{1}{sR^2} \iint_{[(\alpha,\beta),n,R,0]} u(x, y)\,dx\,dy = \sum_{j=0}^{p} B_j \triangle^j u(\alpha, \beta) R^{2j} + o(R^{2p}),$$

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