ON MINIMAL COMPLETELY REGULAR SPACES ASSOCIATED WITH A GIVEN RING OF CONTINUOUS FUNCTIONS

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1. INTRODUCTION

Let C(X) denote the ring of all continuous real-valued functions on a completely regular space X. If X and Y are completely regular spaces such that one is dense in the other, say X is dense in Y, and every $f \in C(X)$ has a (unique) extension $\bar{f} \in C(Y)$, then C(X) and C(Y) are said to be *strictly isomorphic*. In a recent paper [2], L. J. Heider asks if it is possible to associate with the completely regular space X a dense subspace μX minimal with respect to the property that $C(\mu X)$ and C(X) are *strictly isomorphic*.[1]

In this note, Heider's question is answered in the negative. It is shown, moreover, that if μX exists, then it consists of all of the isolated points of X, together with those nonisolated points p of X such that $C(X \sim \{p\})$ and C(X) fail to be strictly isomorphic. Thus, if μX exists, it is unique.

2. PRELIMINARY REMARKS

Let C(X) denote the ring of all continuous real-valued functions on a completely regular space X. Let C*(X) denote the subring of all bounded $f \in C(X)$. The following known facts are utilized below.

(2.1) Corresponding to each completely regular space X, there exists an essentially unique compact space βX, called the Stone-Čech compactification of X, such that (i) X is dense in βX, and (ii) every $f \in C^*(X)$ has a (unique) extension $\bar{f} \in C^*(\beta X) = C(\beta X)$. Thus C*(X) and $C(\beta X)$ are isomorphic. (See, for example, [3] or [4, Chapter 5].)

(2.2) There exists an essentially unique subspace υX of βX such that (i) X is a Q-space, (ii) X is dense in υX, and (iii) every $f \in C(X)$ has a (unique) extension $\bar{f} \in C(\upsilon X)$. Thus C(X) and $C(\upsilon X)$ are isomorphic. (For the definition of Q-space, and a proof of this theorem, see [1] or [3].)

(2.3) If X and Y are completely regular spaces such that C(X) and C(Y) are isomorphic, then Y is homeomorphic to a dense subspace of υX such that every real-valued function continuous on this subspace has a (unique) continuous extension over υX. [3, Theorem 65.]

(2.4) If Z is any compact space, and f is any continuous mapping of X into Z, then there exists a (unique) continuous extension $\hat{f}$ of f over βX into Z. (See [5, Theorem 88].)

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1. Since the writing of this paper, Heider's problem has been generalized and solved independently by J. Daly and L. J. Heider.