INVERSION OF TWO THEOREMS OF ARCHIMEDES

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Archimedes proved the following theorems:

THEOREM I. *The curved surface of a cylinder circumscribed about a sphere is equal to the surface of the sphere.*

THEOREM II. *The volume of a cylinder circumscribed about a sphere is* 3/2 *times that of the sphere.*

We shall invert these theorems as follows:

THEOREM 1. *If the curved surface of each right cylinder circumscribed about a convex body* K *is equal to the surface of* K, *then* K *is a sphere.*

THEOREM 2. *If the volume of each right cylinder circumscribed about a convex body* K *is* 3/2 *times that of* K, *then* K *is a sphere.*

We shall first deal with Theorem 1. Let L be the length of the closed convex curve K which is obtained by an orthogonal projection of K; let B denote the breadth of K in the direction of the projection; and let dw be the solid angle element of the directions of projection. We consider the integral

$$J = \int LB\,dw, \tag{1}$$

extended over the whole unit sphere. It is known that L can be represented by the integral

$$\int_0^{2\pi} p\,d\phi, \tag{2}$$

where p denotes the support function of the projection of K, and where $d\phi$ denotes the angle element of its tangent. Therefore (1) can be written in the form

$$J = \int\left(\int p\,d\phi\right)B\,dw \tag{3}$$

Now we take advantage of the concepts of integral geometry [1]. The symbol $d\phi\,dw$ represents the density of the configuration consisting of one space direction D_1 and a direction D_2 perpendicular to D_1. Since this configuration has no invariant under motions, we have

$$d\phi\,dw = C\,d\bar{\phi}\,d\bar{w} \tag{4}$$

where $d\bar{w}$ is the solid angle element of the normals of the plane $D_1 D_2$, and where $\bar{\phi}$ measures the rotations in this plane. C is easily shown to be 1 (for instance, by integration over the whole unit sphere). Hence, it follows that (3) can be transformed into

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