A THEOREM ABOUT LOCAL BETTI GROUPS

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SUMMARY. The theory of exact sequences is applied to local homology invariants to obtain a relation analogous to the Mayer-Vietoris formula. This formula is then applied to the local Betti numbers in an lc^n space, and it yields a result concerning generalized manifolds with boundary; this result is described in Section 2.

1. The local Betti numbers are basic tools in the theory of generalized manifolds (see Wilder [8], p. 190 ff.) Alexandroff discussed these invariants in an earlier paper, calling them *Betti numbers around a point* ([1], p. 2). In the same paper, the *Betti groups in a point* were defined (p. 19), and the equivalence of r-dimensional local Betti numbers and the rank of the r^{th} Betti group in a point where there is no r-condensation was proved (Corollary to Theorem III, p. 22). Wilder has proved [6] that no point has r-condensation in an lc^n space, for $r \leq n$, and he has recently shown [7] that Alexandroff's equivalence result holds in a much wider class of spaces. In the following, the theorems are stated and proved for Betti groups in a point, and they are then applied to concepts defined in terms of local Betti numbers, the rank of the r^{th} Betti group being equal to the r-dimensional local Betti number when such application is made.

Let S be a locally compact space, X a closed subset of S, A a closed subset of X, and x a point in A. In analogy with the terminology in the large, (x: X, A) is called a local pair. Let $\{P_\alpha\}$ be a basis for the open sets of S containing x. Denote by $Z^n(x: X, X - P_\alpha)$ the vector space of Čech n-cycles on X mod $(X - P_\alpha)$, with coefficients in a field; denote by $B^n(x: X, X - P_\alpha)$ the subspace of $Z^n(x: X, X - P_\alpha)$ which consists of n-cycles that bound on X mod $(X - P_\alpha)$. The quotient of these spaces is denoted by $H^n(x: X, X - P_\alpha)$. Inclusion among sets in the collection $\{P_\alpha\}$ induces an order relation among the indices $\{\alpha\}$; that is, $P_\alpha \subset P_\beta$ implies $\alpha > \beta$, and the direct limit group $\lim_{\rightarrow} H^n(x: X, X - P_\alpha)$ can be defined. This is called the *Betti group in a point*, and it will be denoted $LH^n(x: X)$.

For the study of the Betti groups in a point of a local pair, it is convenient to consider further the limit groups

$$\lim_{\rightarrow} H^n(x: A \cup (X - P_\alpha), X - P_\alpha) = LH^n(x: A),$$

$$\lim_{\rightarrow} H^n(x: X, A \cup (X - P_\alpha)) = LH^n(x: X, A)$$

and the maps

$$i_*: LH^n(x: A) \to LH^n(x: X),$$

$$j_*: LH^n(x: X) \to LH^n(x: X, A),$$

$$\partial : LH^n(x: X, A) \to LH^{n-1}(x: A),$$

which are defined as the maps of the limit groups defined by the inclusion maps and boundaries of the terms.

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