SOME REMARKS ON ELEMENTARY DIVISOR RINGS II

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1. INTRODUCTION

A commutative ring S with identity element 1 is called an *elementary divisor ring* (resp. *Hermite ring*) if for every matrix A over S there exist nonsingular matrices P, Q such that PAQ (resp. AQ) is a diagonal matrix (resp. triangular matrix). It is clear that every elementary divisor ring is an Hermite ring, and that every Hermite ring is an F-ring (that is, a commutative ring with identity in which all finitely generated ideals are principal).

We are concerned, in this paper, with identifying those F-rings that are elementary divisor rings. It is known that every F-ring satisfying the ascending chain conditions on ideals is an elementary divisor ring [5, Theorem 12.3, ff]. The earliest affirmative result of this kind obtained without chain conditions is Helmer's result that every *adequate ring* without (proper) divisors of 0 is an elementary divisor ring [3]. (An F-ring S is an *adequate ring* if, for every $a, b \in S$ with $a \neq 0$, we may write $a = rs$ with $(r, b) = (1)$ and with $(t, b) \neq (1)$ for every nonunit divisor t of s. As usual, $(a_1, \cdots, a_n)$ denotes the ideal generated by $a_1, \cdots, a_n$.) This result was generalized successively by Kaplansky [5, Theorem 5.3] and by L. Gillman and the author [1]. The latter showed that every adequate Hermite ring is an elementary divisor ring, and they gave examples [2, Corollary 6.7] of elementary divisor rings that are not adequate rings. In addition they gave examples of F-rings that are not Hermite rings, and of Hermite rings that are not elementary divisor rings [2, Examples 3.4 and 4.11]. All of these examples have divisors of 0. We give below what seems to be the first known example of an elementary divisor ring without divisors of 0 that is not an adequate ring.

In addition, by using theorems in [1] and [5], we obtain the following affirmative results: 1. If the Perlis-Jacobson radical [4] R(S) of the F-ring S contains a prime ideal of S, then S is an Hermite ring. 2. In order that an Hermite ring S be an elementary divisor ring, it is enough that S/R(S) be an elementary divisor ring. 3. Every nonzero (proper) prime ideal of an adequate ring S is contained in a unique maximal ideal of S. 4. If S is an Hermite ring and every element of S not in R(S) is contained in at most a finite number of maximal ideals, then S is an elementary divisor ring.

In the last section of the paper, we give the example cited above and state some unsolved problems.

2. THE AFFIRMATIVE RESULTS

The following theorem, which is proved in [1], is used repeatedly below.

THEOREM 1. (a) *A commutative ring with identity is an Hermite ring if and only if it satisfies the condition* (T): *for all* $a, b \in S$, *there exist* $a_1, b_1, d \in S$ *such that* $a = a_1 d$, $b = b_1 d$, *and* $(a_1, b_1) = (1)$.

Received April 13, 1956.

The preparation of this paper was sponsored (in part) by the National Science Foundation, under grant NSF G1129.