THE POINT SPECTRUM OF WEAKLY ALMOST PERIODIC FUNCTIONS

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1. INTRODUCTION

We adopt the terminology and notation of our first paper [1] on the family $\mathfrak{W}$ of weakly almost periodic functions on a locally compact Abelian group G. With every w.a.p. function x(t) we associate a formal Fourier series

$$\sum_{\lambda \in G^*} a(\lambda)(t, \lambda),$$

where $a(\lambda) = M_s[x(s)(-s, \lambda)]$. We show that the Fourier series is the Fourier series of an almost periodic (a.p.) function $x_1(t)$; that is, every w.a.p. function x(t) admits a unique decomposition $x = x_1 + x_2$, where x_1 is a.p. and $M(|x_2|^2) = 0$. The set $[\lambda : a(\lambda) \neq 0]$ becomes the discrete or discontinuous part of the spectrum $\sigma(x)$ (see [2]).

The basic ergodic theorem which underlies the mean value theory of w.a.p. functions in [1] now reappears in the guise of a summability theorem.

2. ABSTRACT SUMMABILITY THEORY

In the present context, summability theory rests on the almost periodic properties of the kernel:

LEMMA 1. *Let* x *be w.a.p., and let* y *be a.p. with the properties* $y(t) > 0$, $y(-t) = y(t)$, $M(y) = 1$. *Then* $x * y$ *lies in* $\bar{O}(x)$, *the closed convex hull of the translates of* x.

Proof. For every t,

$$(x*y)(t) = M_s[x(s)y(t-s)] = M_s[x(s)y(s-t)] = \lim_{\alpha} T_\alpha(xy_t),$$

where the T_α run through the semi-group of finite convex combinations of translation operators $x(s) \to x_u(s) = x(s-u)$ ordered by multiplication (see [1], p. 225). Since the T_α are equi-uniformly continuous and the set $\{y_t\}$ and (hence) the set $\{x \cdot y_t\}$ $(t \in G)$ are conditionally compact in the norm topology of C(G), the convergence is uniform in t. It follows that for every $\varepsilon > 0$ there exists a finite set $\{s_n\}$ in G and a set $\{a_n\}$ of positive real numbers with $\Sigma a_n = 1$ such that simultaneously

$$(1) \qquad |(x*y)(t) - \Sigma a_n x(s-s_n) y(s-s_n-t)| < \varepsilon/2 \qquad (s, t \in G),$$

$$(2) \qquad \Sigma a_n y(-s_n) = b, \quad \text{where } |1 - 1/b| < \frac{\varepsilon}{2\,\|x\| \cdot \|y\|}.$$

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