A NOTE ON POWER SERIES AND AREA

F. Bagemihl

Let C denote the unit circle $|z| = 1$, and D the open unit disk $|z| < 1$ in the complex plane. We shall find it convenient to refer to the intersection of D with a neighborhood of a point $e^{i\theta} \in C$ as a "neighborhood of $e^{i\theta}$." If $f(z)$ is a holomorphic function in D, we call $e^{i\theta}$ a *strong point* of $f(z)$ provided that every neighborhood of $e^{i\theta}$ is mapped by $f(z)$ onto a Riemann configuration of infinite area; otherwise we call $e^{i\theta}$ a *weak point* of $f(z)$. Every strong point of $f(z)$ is obviously also a singular point of this function. The converse, however, is not true, for if $f(z)$ is schlicht and maps D onto a region bounded by a Jordan curve possessing no analytic subarc, then every $e^{i\theta}$ is a singular point as well as a weak point of $f(z)$.

We are going to modify a result (and its proof) due to Ryll-Nardzewski and Steinhaus [5; 1, p. 102] to show that as a rule (in a certain sense) $f(z)$ has every $e^{i\theta}$ for a strong point. We are indebted to W. Seidel for some helpful suggestions.

THEOREM. *For every* x *in some Banach space* X, *let* $f(x, z)$ *be a holomorphic function of* $z \in D$, *and for every* $z \in D$, *let* $f(x, z)$ *be a linear functional of* $x \in X$. *Then there exists an open set* $G \subset C$, *and a set* $Q \subset X$, *of type* F_σ *and of first category, such that, for every* $x \in X$, *every* $e^{i\theta} \in G$ *is a weak point of* $f(x, z)$, *and, for every* $x \in X - Q$ *(a residual subset of* X*), every* $e^{i\theta} \in C - G$ *is a strong point of* $f(x, z)$.

If, further, to every $e^{i\theta} \in C$ *there corresponds an* $x_\theta \in X$ *such that* $e^{i\theta}$ *is a strong point of* $f(x_\theta, z)$, *then, for every* $x \in X - Q$, *every* $e^{i\theta} \in C$ *is a strong point of* $f(x, z)$.

Before proving the theorem, we consider the following example. Let

$$f(x, z) = \sum_{k=0}^{\infty} a_k z^k, \quad x = \{a_k\}, \quad a_k \text{ complex} \quad (k = 0, 1, 2, \cdots),$$

and take X to be the Banach space which consists of all bounded sequences x, with $\|\{a_k\}\| = \sup_k |a_k|$ (this is the space X_2 of Ryll-Nardzewski and Steinhaus [5; 1, p. 104]). According to Lusin [3; 2, p. 69], there exists a power series $\sum b_k z^k$, with $\lim_{k\to\infty} b_k = 0$, which diverges at every point of C. If we set $\beta = \{b_k\}$ $(k = 0, 1, 2, \cdots)$, then $\beta \in X$, and it follows from a result of Zygmund [7] that every point $e^{i\theta}$ is a strong point of $f(\beta, z)$. Consequently, according to our theorem, there exists a residual set $R \subset X$ such that, for every $x \in R$, $f(x, z)$ maps every neighborhood of every point of C onto a Riemann configuration of infinite area.

Proof of the theorem. By a *rational arc* we mean an open subarc of C whose end points have principal amplitudes that are rational numbers. We call a rational arc A a *weak arc* provided that, for every $x \in X$, every $e^{i\theta} \in A$ is a weak point of $f(x, z)$. Denote the set of all weak arcs by W, let G be the union of all weak arcs, and set $H = C - G$. Then obviously, for every $x \in X$, every $e^{i\theta} \in G$ is a weak point of $f(x, z)$.

For every natural number n and every rational arc A, we denote the region

Received November 25, 1955.