A NOTE ON TOPOLOGIES ON $2^{\mathfrak{R}}$

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In this paper we consider topologies on $2^{\mathfrak{R}}$, the set of functions on $\mathfrak{R}$ into the two-element set $\{0, 1\}$. We make no distinction between $2^{\mathfrak{R}}$ and the collection of subsets of $\mathfrak{R}$; that is, we identify each subset of $\mathfrak{R}$ with its characteristic function.

We define the *sequential order topology* τ_1 as follows: A subset B of $2^{\mathfrak{R}}$ is closed if it contains the limit of each convergent sequence contained in B; the sequence $\{x_n\}$ is said to converge to x whenever

$$\bigcup_m \bigcap_{n \geq m} x_n = x = \bigcap_m \bigcup_{n \geq m} x_n$$

(see [1]). The *transfinite sequential order topology* τ_2 and the *order topology* τ_3 are defined analogously, with "sequence" replaced by "transfinite sequence" for τ_2 and by "net" for τ_3. By "transfinite sequence" we mean a function on a well ordered set into a set, and by "net" we mean a function on a directed set into a set. The topology of pointwise convergence is the usual strong product topology where the discrete topology is taken as the topology for $\{0, 1\}$. We denote this topology by τ_4.

Notation. For convergence under τ_i (i = 1, 2, 3, 4), we write $x_\alpha \to x$ under τ_i. If x is a point of accumulation under τ_i of a set A, we write $x \in A'$ (τ_i). As usual, the statement that $x \in A'$ (τ_i) means that every set open under τ_i and containing x contains points of A - (x). We order topologies as follows: τ is stronger than or equal to τ' $(\tau \geq \tau')$ if each set open (closed) under τ is also open (closed) under τ'. For the characteristic function f_x of a subset x of $\mathfrak{R}$, and for j in $\mathfrak{R}$, we mean by the statements $f_{x_n}(j) \to f_x(j)$ and $f_{x_\alpha}(j) \to f_x(j)$ that the sequence $\{f_{x_n}(j)\}$ in $\{0, 1\}$ converges to $f_x(j)$ and that the net $\{f_{x_\alpha}(j)\}$ converges, as a Moore-Smith limit, to $f_x(j)$, respectively. By $2^{\mathfrak{R}}$ (τ) we mean the space determined by the set $2^{\mathfrak{R}}$ with topology τ.

THEOREM 1. *Every projection* π_j $(j \in \mathfrak{R})$ *of* $2^{\mathfrak{R}}$ *is continuous under* τ_i (i = 1, 2, 3, 4). *Also,* $\tau_1 \leq \tau_2 \leq \tau_3 \leq \tau_4$.

Proof. For τ_4 the result is well known and is an immediate consequence of the definition of τ_4. For the other cases we note that $\pi_j^{-1}(1)$ is the collection of subsets of $\mathfrak{R}$ which contain j. It is both open and closed under τ_1, τ_2 and τ_3; and $\pi_j^{-1}(0)$ is the complement of $\pi_j^{-1}(1)$. This shows that τ_i (i = 1, 2, 3) is weaker than or equal to τ_4, since τ_4 is the strongest topology under which every projection is continuous. Since every sequence is a transfinite sequence and every transfinite sequence is a net, we have $\tau_1 \leq \tau_2 \leq \tau_3$. This implies the ordering $\tau_1 \leq \tau_2 \leq \tau_3 \leq \tau_4$.

THEOREM 2. *If* $\mathfrak{R}$ *is countable, then* $\tau_1 = \tau_2 = \tau_3 = \tau_4$.

Proof. Suppose $x \in A'$ (τ_4). Then, since τ_4 is metric, $\mathfrak{R}$ being countable, there exists a sequence $\{x_n\}$ in A - (x) such that $x_n \to x$ under τ_4. But this means that $f_{x_n}(j) \to f_x(j)$ for each j in $\mathfrak{R}$, where f_{x_n} and f_x are the characteristic functions

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