THE DISTRIBUTION OF VALUES OF MULTIPLICATIVE FUNCTIONS

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1. INTRODUCTION. E. Landau [1] has shown that, as $x \to \infty$, the number of positive integers not exceeding x which are representable as the sum of two squares is asymptotic to

$$Bx(\log x)^{-1/2}, \tag{1}$$

where

$$B = \left(2^{-1} \prod_{p \equiv 3 (\mathrm{mod}\ 4)} \frac{1}{1 - p^{-2}}\right)^{1/2}.$$

P. Lévy [2] gave a simple heuristic derivation of (1) without determining B; his argument also led him to the conjecture that, if $r_2(m)$ is the number of representations of m as the sum of the squares of two positive integers, and $R_k(x)$ is the number of $m \leq x$ for which $r_2(m) = k$, then

$$R_k(x) \sim \frac{Bx}{\log^{1/2} x} \cdot \frac{e^{-\theta}\theta^k}{k!}, \quad \text{where } \theta = c \log^{1/2} x.$$

In probabilistic terms, this means roughly that, out of the integers m for which $r_2(m) > 0$, those for which $r_2(m)$ has a specified value have a Poisson distribution, with parameter θ.

It will be shown here that this is not the case, and that in fact the asymptotic behavior of $R_k(x)$ depends rather strongly on the arithmetic structure of k, as well as on its size. This is not very surprising, since r_2, being a multiplicative function, must be considered in probability language as a product of random variables, while the usual theory applies to sums of random variables. Thus A. Wintner [3] has shown that if f is an additive function [so that $f(mn) = f(m) + f(n)$ whenever $(m, n) = 1$] with the property that $f(p) = 1$ and $f(p^\alpha) > 0$ for all primes p and all $\alpha > 1$, then the number of solutions of $f(m) = k$ which do not exceed x is asymptotic to

$$\frac{x(\log\log x)^{k-1}}{(k-1)!\log x};$$

this is a "Poisson distribution" with parameter $\log\log x$. In particular, if $\omega(m)$ is the total number of prime divisors of m, $\tau(m)$ is the number of divisors of m, and $f(m) = \omega(\tau(m))$, then f satisfies Wintner's hypotheses, so that the integers m for which $\tau(m)$ has a specified number of prime factors are Poisson distributed; that this is not true of the m for which $\tau(m)$ has a specified value is shown in §2.

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