ON THE SCHWARZ REFLECTION PRINCIPLE

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Let $f(z)$ be meromorphic in the unit circle K: $|z| < 1$, and let the modulus $|f(re^{i\theta})|$, $z = re^{i\theta}$, have radial limit 1 for almost all $e^{i\theta}$ belonging to some arc A: $0 \leq \theta_1 < \theta < \theta_2 \leq 2\pi$ of $|z| = 1$. We are interested in conditions under which $f(z)$ may be continued analytically across the arc A by means of the functional relation $f(1/\bar{z}) = 1/\overline{f(z)}$. Using a concept originating with Gross [4], we associate with an arbitrary point P of $|z| = 1$ three sets of points: the *cluster set* $C(P)$ of $f(z)$ at P, defined as the set of all values α which $f(z)$ approaches on a sequence of points of K converging to P; the *range of values* $R(P)$ of $f(z)$ at P, which consists of all the values α which $f(z)$ assumes infinitely often in every neighborhood of P; the *asymptotic set* $\Gamma(P)$ of $f(z)$ at P, which consists of all values α which $f(z)$ approaches along a Jordan arc which lies, except for one endpoint, entirely in K, and which terminates at the point P. In this connection we shall say that a value α in $\Gamma(P)$ is an *asymptotic value* of $f(z)$ at P.

The most interesting problem is that of the behavior of $f(z)$ in the neighborhood of P whenever P is a singular point of A; that is, whenever $f(z)$ cannot be continued analytically across any arc of $|z| = 1$ containing P. Among the most significant results concerning the behavior of meromorphic functions in the neighborhood of a singular point on $|z| = 1$, we mention the recent theorem of Carathéodory [2; 266] which states that the cluster set $C(P)$ for each singular point P on A is either one of the two sets $|w| \leq 1$, $|w| \geq 1$, or else the extended plane. Nevanlinna [8; 28] had previously shown that if $f(z)$ is analytic and bounded, $|f(z)| < 1$, in $|z| < 1$, and if $\lim_{r\to 1} |f(re^{i\theta})| = 1$ almost everywhere on an arc A of $|z| = 1$, then the radial limit values $f^*(e^{i\theta})$ of $f(z)$ in any neighborhood of a singular point P on A comprise a set on $|w| = 1$ of measure 2π. This result was improved by Seidel [11; 208], who showed that every point of $|w| = 1$ is a radial limit value of $f(z)$ in any neighborhood of P.

Our principal result, previously announced in [5], is that a meromorphic function having at most a finite number of zeros and poles in the region $0 < 1 - \varepsilon < |z| < 1$, $\theta_1 < \theta < \theta_2$, can be continued analytically beyond the arc A: $\theta_1 < \theta < \theta_2$ by means of the Schwarz reflection principle if and only if $f(z)$ admits neither 0 nor ∞ as an asymptotic value at any point of A. From this result follows an extension of the theorems of Nevanlinna and Seidel to the case of meromorphic functions of bounded characteristic. We prove first a theorem for meromorphic functions of bounded characteristic which extends a result of Seidel [11; 207] for bounded functions, and which provides the motivation for Theorem 2.

THEOREM 1. *Let* $f(z)$ *be meromorphic with bounded characteristic in* $|z| < 1$, *and let* $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ *exist with modulus* 1 *for almost all* $e^{i\theta}$ *belonging to an arc* A: $0 \leq \theta_1 < \theta < \theta_2 \leq 2\pi$ *of* $|z| = 1$. *If* $f(z)$ *has no zeros or poles in the region* $0 < 1 - \varepsilon < |z| < 1$, $\theta_1 < \theta < \theta_2$, *then the set of singularities of* $f(z)$ *on* A *is the closure on* A *of the set of points* $e^{i\theta}$ *for which* $f^*(e^{i\theta}) = 0$ *or* $f^*(e^{i\theta}) = \infty$.

As a function of bounded characteristic, $f(z)$ has a representation (see, e.g., Nevanlinna [7; 190])

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