ON FABER SERIES

1. A PROBLEM OF TRANSFER

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1. INTRODUCTION. In Sections 2 and 3 a method for the treatment of Faber series [2] is developed. The method is applied, in Section 4, to give a new proof of a recent result of Iliev [4], and to establish one new theorem. Further applications are indicated in Section 5.

1.1. *Notation.* The letter C will denote the same simple closed analytic curve throughout, and I(C) will denote its interior. The symbol F(z) will represent a function analytic in I(C), although not necessarily the same one in different usages. The symbol Σ will indicate a summation in which the index of the summand ranges from 0 to ∞. A sequence will be represented by placing braces about a general element. Again, the index will be understood to range from 0 to ∞.

1.2. *The Problem of Transfer.* The following proposition constitutes the basic result in the theory of Faber series.

LEMMA 1 (Faber). *There exists a sequence of polynomials* $\{F_n(z)\}$ *which can be associated with* C, *such that each function analytic in* I(C) *can be represented by a unique series*

$$\Sigma\, a_n\, F_n(z) \tag{1}$$

converging uniformly in each closed subset of I(C).

These polynomials are now called Faber polynomials. A series of type (1), whether it converges for any z, or not, is called a Faber series. When the series converges uniformly in closed subsets of I(C), it converges to an analytic function and is called the Faber expansion of the function.

Recently, Iliev investigated the nature of the analytic function represented by a Faber series in the case where the number of different values assumed by the coefficients is finite. The corresponding problem for power series was solved by Szegö [6]. He proved that the function represented by the power series is either a rational function, or is analytic inside the circle $|z| = 1$ and has each point of this circle as a singularity.

Iliev's result has a similar character, and his proof follows the pattern of the proof for power series. This suggests the *problem of developing a general method for transferring a theorem on power series to Faber series.* As indicated, Iliev's proof is of a special nature. In Faber's work, however, certain findings are related to this problem. It will be indicated in what respect they are inadequate as a method of transfer. A new lemma is then added, which, together with Faber's results, constitutes the proposed method.

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