EXTENSIONS OF THE GROSS STAR THEOREM

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1. BACKGROUND. The Gross star theorem ([3]; [5], p. 276) asserts that each element $z(w)$ of the inverse of a function $w = \phi(z)$, meromorphic for $|z| < \infty$, can be continued to infinity along almost all rays from the center of the element. This has been generalized [4]: first, by replacement of the rays by very general families of "parallel curves"; and second, by replacement of the class of inverses of meromorphic functions by a considerably broader class. The generalizations depended on the following theorem concerning schlicht functions ([4], p. 4):

THEOREM I. *Let* $t = h(s)$ *be lower semi-continuous for* $0 < s < 1$, *where* $0 < b \leq h(s) \leq +\infty$. *Let* $w = \psi(\sigma)$ $(\sigma = s + it)$ *be schlicht in the domain* G: $0 < s < 1$, $-b < t < h(s)$; *and let*

$$\lim_{t \to h(s)} \psi(s + it) = 0, \quad h(s) < \infty \tag{1}$$

for each s *in a subset* E *of* (0,1). *Then* E *has measure* 0.

In the same paper ([4], p. 20) the following theorem was proved:

THEOREM II. *Let* E *be a closed set of capacity zero on* $|z| = 1$. *Then there exists a schlicht function* $w = \Phi(z)$ *in* $|z| < 1$ *such that* $\lim_{z \to z_0} \Phi(z) = \infty$ *for each* z_0 *in* E, *while* $\lim_{z \to z_1} \Phi(z)$ *is finite for* $|z_1| = 1$ *and* z_1 *not in* E.

2. TWO THEOREMS ON SCHLICHT FUNCTIONS.

THEOREM 1. *Let* B *be a closed countable subset of the extended plane. In Theorem* I *let* (1) *be replaced by the condition*

$$\lim_{n \to \infty} \psi(s + it_n) \in B, \quad h(s) < \infty \tag{1'}$$

for every sequence $t_n \to h(s)$, *whenever* $\lim \psi(s + it_n)$ *exists or is* ∞. *Then the conclusion that* E *has measure zero remains valid.*

Proof. The limits in (1') are the "cluster values" of ψ on the segment $s = \text{const.}$, $-b < t < h(s)$, as t approaches the boundary $h(s)$. These cluster values must form a closed connected set. However, B is totally disconnected. Hence, for each s in E the limit in (1') exists for all sequences $t_n \to h(s)$; that is, $\lim_{t \to h(s)} \psi(s + it)$ exists or is ∞ and is an element b_k of B $(k = 1, 2, \cdots)$. For each b_k, let E_k be the subset of E for which the limit equals b_k. Then E_k has measure zero, by Theorem I, so that $E = \bigcup_k E_k$ has measure zero.

Remark 1. It is natural to conjecture that the theorem remains true if B is an arbitrary totally disconnected closed set. It is certainly false in this generality; for we can choose E to be closed, of positive measure and totally disconnected, h(s) to be 1 and ψ to be the identity. However, it may remain true if B has linear measure zero or if B has capacity zero. The following theorem is a result in this direction.

THEOREM 2. *In Theorem* I *let the function* $z = \psi(\sigma)$ *satisfy the additional hypothesis*: $|\psi(\sigma)| < 1$ *in* G. *Let* B *be a closed subset of* $|z| = 1$ *having capacity*

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