A PROJECTION OPERATOR ON HARMONIC MAPPINGS

by

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1. INTRODUCTION. Throughout this paper, D will denote a simply connected domain in the xy-plane. Let $u = u(x, y)$ and $v = v(x, y)$ be a pair of real-valued functions with continuous second partial derivatives on D; and let w denote the mapping of D into the uv-plane which is defined by these functions. The Jacobian matrix of w,

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

satisfies the matrix differential equation

$$J_x e_2 - J_y e_1 = 0, \tag{1}$$

where e_1 and e_2 are the unit vectors $\binom{1}{0}$ and $\binom{0}{1}$, and where the subscripts indicate that each element of J has been replaced by the corresponding partial derivative. If w is a harmonic mapping, then the further equation

$$J_y e_2 + J_x e_1 = 0$$

is also satisfied. With the notation $\xi = Je_2$ and $\eta = Je_1$, equations (1) and (2) can be written as the vector differential equations

$$\xi_x = \eta_y, \quad \xi_y = -\eta_x,$$

formally similar to the Cauchy-Riemann equations.

It should be noted that if a two-by-two matrix J of differentiable functions satisfies equation (1), it is the Jacobian matrix of a mapping.

2. DEFINITION OF PROJECTION OPERATOR. Henceforth, J will denote the Jacobian matrix of a harmonic mapping w with the components u and v. The operator P will be defined by the relation

$$\hat{J} = P[J] = (\hat{J} + KJK^{-1})/2, \text{ where } K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

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