UPPER AND LOWER BOUNDS OF ORDER TYPES

by

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1. In 1940, Fraissé [1] defined the relation

$$\alpha \leqq \beta$$

to mean that an ordered set A of type α is similar to a subset of an ordered set B of type β. If, at the same time, B is not similar to any subset of A, then we shall write $\alpha < \beta$. It is obvious that this definition depends only on the order types α and β, and is independent of the special sets A and B. If $\alpha \leqq \beta$ and $\beta \leqq \alpha$ both hold, we shall write $\alpha \equiv \beta$ and say that α and β are equivalent (even though α and β may be distinct). If neither $\alpha \leqq \beta$ nor $\beta \leqq \alpha$ holds, then α and β will be said to be non-comparable.

In terms of these relations it is natural to discuss the notions of upper and lower bounds of two order types, or their least upper and greatest lower bounds. Thus, γ would be called a least upper bound for α and β if $\alpha \leqq \gamma$, $\beta \leqq \gamma$, while for any δ such that $\alpha \leqq \delta$ and $\beta \leqq \delta$ it would follow that either $\gamma < \delta$ or that γ and δ are non-comparable.

2. Throughout this note we shall assume as known the usual terminology and symbolism for order-types and ordinals.

The purpose of this note is to give a method for demonstrating the following theorem:

If $\alpha = \omega \cdot r + m$, $\beta = \omega \cdot s + n$, where r and s are natural numbers and m and n are integers $\geqq 0$, then α and β^* have only a finite number of distinct least upper bounds, namely, all types of the form

$$\text{(I)} \qquad n + \omega^* \cdot b_1 + \omega a_1 + \dots + \omega^* \cdot b_t + \omega \cdot a_t + m$$

where t and the coefficients $a_1, \dots, a_t, b_1, \dots, b_t$ are natural numbers except that b_1 or a_t may be 0, and

$$\sum_{i=1}^{t} a_i = r, \qquad \sum_{i=1}^{t} b_i = s.$$

We do not actually prove this theorem; however its proof would be only a slight modification of the proof of Theorem VI in section 4.

Hereafter, we shall call the types (I), with $m = n = 0$, the mixed sums of α and β^*; similarly, an order-type γ will be called a mixed sum if γ can be represented in the form (I) for some ordinals α and β.

3. We first prove a number of auxiliary theorems about mixed sums and their relation to general order types.