ON A THEOREM OF HENRY BLUMBERG

by

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The theorem [1] asserts that for every real function $f(x)$ on $I = (0, 1)$ there is an everywhere dense set E such that $f(x)$ is continuous on E relative to E.

It seems plausible that an analogous result should hold for one-one transformations; i.e., that for every one-one correspondence $f(x)$, $f^{-1}(y)$ between $I = (0, 1)$ and $J = (0, 1)$ there should be sets E and $f(E)$, everywhere dense in I and J, respectively, such that $f(x)$, $f^{-1}(y)$ is a homeomorphism between them. The purpose of this note is to show that this is not true:

<u>There is a one-one correspondence</u> $f(x)$, $f^{-1}(y)$ <u>between</u> $I = (0, 1)$ <u>and</u> $J = (0, 1)$ <u>such that, for every</u> E <u>which is everywhere dense in</u> I, $f(x)$, $f^{-1}(y)$ <u>is not a homeomorphism between</u> E <u>and</u> $f(E)$.

Consider the following two sequences of subintervals of I. For every positive integer n and every $m = 0, 1, \ldots, 2^n - 1$, let $I_{nm} = (\frac{m}{2^n}, \frac{m+1}{2^n})$.

Let $0 < a_1 < \cdots < a_k < \cdots < 1$ be an increasing sequence of positive numbers which converges to 1. Label the semi-open intervals $(0, a_1], (a_1, a_2], \cdots, (a_n, a_{n+1}], \cdots$ as $\tilde{I}_{11} = (0, a_1]$, $\tilde{I}_{12} = (a_1, a_2]$, $\tilde{I}_{21} = (a_2, a_3]$, $\cdots$, so that there is an $\tilde{I}_{nm}$ for every positive n and $m = 0, 1, \cdots, 2^n - 1$. The intervals $\tilde{I}_{nm}$ are mutually disjoint, their union is I, and if $n_1 > n_2$ or if $n_1 = n_2$, $m_1 > m_2$ then $\tilde{I}_{n_1 m_1}$ is to the right of $\tilde{I}_{n_2 m_2}$. Consider also the two sequences J_{nm} and $\tilde{J}_{nm}$ of subintervals of J obtained in the same way..

Now, for each positive n and $m = 0, 1, \cdots, 2^n - 1$, let $S_{nm} \subset I_{nm}$ and $T_{nm} \subset J_{nm}$ be non-empty, perfect, nowhere dense sets such that the sets S_{nm} are mutually disjoint and the sets T_{nm} are mutually disjoint. Let $S = \bigcup S_{nm}$ and $T = \bigcup T_{nm}$. Observe that both S and its complement intersect every subinterval of I in a set of cardinal number c, and that T has the same property relative to J. For each n and m, let $\tilde{S}_{nm} = \tilde{I}_{nm} - S$, $\tilde{T}_{nm} = \tilde{J}_{nm} - T$, and let $\tilde{S} = \bigcup \tilde{S}_{nm}$ and $\tilde{T} = \bigcup \tilde{T}_{nm}$.

It is clear that the $\tilde{S}_{nm}$, as well as the $\tilde{T}_{nm}$, are mutually disjoint, that $I = S \cup \tilde{S}$, $J = T \cup \tilde{T}$, and that every S_{nm}, $\tilde{S}_{nm}$, T_{nm} and $\tilde{T}_{nm}$ has cardinal number c. The correspondence $f(x)$, $f^{-1}(y)$ is defined by means of arbitrary one-one correspondences between S_{nm} and $\tilde{T}_{nm}$ and between $\tilde{S}_{nm}$ and T_{nm} for every n and m.