ON BIORTHOGONAL SYSTEMS

by

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1. The primary purpose of this paper is a didactic one: we want to present the theory of biorthogonal systems in a more general and systematic way than it has been done before; most of our results are easy generalizations of known theorems, especially of recent work on bases in Banach spaces (see [2], [8], [10], [12], [13], [15]). The only special feature of our treatment consists in laying more emphasis on the weak topologies than is usually done, and this proves to be the unifying principle of the theory[1].

2. Let F and G be two vector spaces (over the real or the complex number field) in duality [6]. A system consisting of a family $(a_\lambda)_{\lambda \in L}$ of points of F and a family $(b_\lambda)_{\lambda \in L}$ of points of G is said to constitute a biorthogonal system if $\langle a_\lambda, b_\lambda \rangle = 1$ for all $\lambda \in L$ and $\langle a_\lambda, b_\mu \rangle = 0$ for $\lambda \neq \mu$.

PROPOSITION 1. *Let $(a_\lambda)_{\lambda \in L}$ be a family of points of F. In order that there exist in G a family $(b_\lambda)_{\lambda \in L}$ forming with (a_λ) a biorthogonal system, it is necessary and sufficient that (a_λ) be topologically free for the topology $\sigma(F, G)$* (that is, for every $\lambda \in L$, a_λ does not belong to the closed subspace generated by the a_μ of index $\mu \neq \lambda$; see [4, p. 24]). *Moreover if (b_λ) and (b'_λ) are two such families, then $b'_\lambda - b_\lambda \in A^o$, where A is the closed subspace of F generated by the family (a_λ); in particular, $b'_\lambda = b_\lambda$ for all $\lambda \in L$ if and only if $A = F$.*

The proof is an easy application of Hahn-Banach's theorem, and will therefore be omitted.

3. A biorthogonal system $(c_\mu)_{\mu \in M}$, $(d_\mu)_{\mu \in M}$ $(c_\mu \in F,\ d_\mu \in G)$ is said to be an extension of a biorthogonal system $(a_\lambda)_{\lambda \in L}$, $(b_\lambda)_{\lambda \in L}$ if $L \subset M$, and $a_\lambda = c_\lambda$, $b_\lambda = d_\lambda$ for $\lambda \in L$. A biorthogonal system (a_λ), (b_λ) is maximal if it has no proper extension. From Zorn's lemma it follows immediately that

PROPOSITION 2. *Every biorthogonal system has a maximal extension.*

Maximal biorthogonal systems are characterized by the following property:

PROPOSITION 3. *Let (a_λ), (b_λ) be a biorthogonal system, A the closed subspace* (for $\sigma(F, G)$) *generated by (a_λ), B the closed subspace*

(1) We are following the terminology and notations of [4], [5], [6] and [7].